NAME:_____________**KEY**________________________ **ALPHA NUM:**___________
INSTRUCTOR: ____________________________ **SECTION:** ______________

CALCULUS I (SM121, SM121A, SM131) **FINAL EXAMINATION**

0755-1055 Friday 18 December 2009 SHOW ALL WORK IN THIS TEST PACKAGE

Tear off the last sheet of this exam. It contains problems 27, 28, 29, 30. Do these problems first WITHOUT YOUR CALCULATOR. When you are finished with these 4 problems, return the sheet to your instructor, take out your calculator, and complete the rest of the exam.

PART ONE: MULTIPLE CHOICE (50%). The first 20 problems are multiple-choice. Fill in the best answer on your Scantron bubble sheet. Write your name, alpha number, instructor, and section number on this test and your bubble sheet and bubble in your alpha number. There is no extra penalty for wrong answers on the multiple choice. Show all your scratch work on this test.

CALCULATORS ARE PERMITTED FOR THIS MULTIPLE CHOICE SECTION.

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1. What is the DOMAIN of the function given by: $h(x) = \frac{1}{\sqrt{4-x^2}}$

a) $x \neq 2$ b) all reals c) $[-2,2]$ [d)] [$(-2,2)$] e) $(0,\infty)$

[d] since can't take square root of negative and can't divide by 0

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2. Which of the following equations has the graph of an EVEN function?

a) $y = \sin(x)$ [b)] [$y = \cos(x)$] c) $y = e^x$ d) $y = \ln(x)$ e) $y = 1/x$

[b] since $\cos(-x) = \cos(x)$ (also, graph is symmetric across the *y*-axis.

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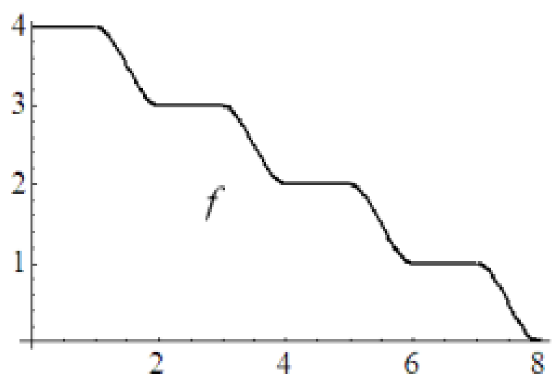
3. For f the function graphed and $g(x) = e^x$, find the value of the composition at 1, namely, $(f \circ g)(1) = f(g(1)) =$

a) 0 b) 1 c) 2 [d) 3] e) e^4



[d] since $e \cong 2.7$, $g(1) = e$ is between 2 and 3 and from the graph $(f \circ g)(1) = f(e) = 3$.

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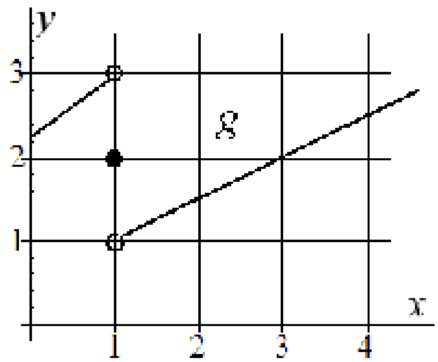
4. If $\lim_{x\to a} f(x) = 3$ and $\lim_{x\to a} g(x) = 0$ and $\lim_{x\to a} h(x) = 1$, then $\lim_{x\to a}(f(x)g(x) + h(x)) =$

a) 0 [b) 1] c) 2 d) 3 e) not enough information to evaluate

[b] since $\lim_{x\to a}(f(x)g(x)+h(x)) = \lim_{x\to a}(f(x)) \lim_{x\to a}(g(x)) + \lim_{x\to a}(h(x)) = 3 \cdot 0 + 1 = 1$

5. Based on the given graph of g, the left-hand limit as x approaches 1 from the left hand side is given by

$\lim_{x \to 1^-} g(x) =$



a) 0 b) 1 c) 2 d) 3 e) does not exist

d since for x close to, but less than, 1, $g(x)$ is close to 3.

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6. Which equation is of a horizontal asymptote to the graph of

$y = \frac{2x^2+1}{x^2-4x+3}$

a) $x = 1$ b) $x = 3$ c) $y = 0$ d) $y = 1$ e) $y = 2$

e since $\lim_{x\to\pm\infty} \frac{2x^2+1}{x^2-4x+3} = \lim_{x\to\pm\infty} \frac{2+(\frac{1}{x^2})}{1-(\frac{4}{x})+(\frac{3}{x^2})} = 2$

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7. Suppose that the table shows the percentage P of the population in Europe that uses cell phones in each year.

Year	2000	2001	2002	2003
P	52	61	73	82

What is the average rate of cell phone growth from 2000 to 2003? (Units are percent/yr.)

a) 10 b) 15 c) 30 d) 82 e) none of these

a since average rate = (change of P)/(change of Year)=$\frac{82-52}{2003-2000} = 10$

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8. Which is an equation of the tangent line to $y = x^2 - 2x^3$ at $(-1,3)$?

a) $y - 3 = (2x - 6x^2)(x+1)$ b) $y - 3 = -8(x+1)$ c) $y = -8x$

d) $y + 1 = -8(x-3)$ e) none of these

b since $y' = 2x - 6x^2$, so at $(-1,3)$ slope is given by $m = -2 - 6 = -8$. Point-slope form of the line gives b.

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9. If $f(4) = 2$ and $f'(4) = 3$ and $g(x) = \sqrt{x}$, then the derivative of the product at 4 is $(fg)'(4) =$

a) 4 b) 6 c) 8 d) $\frac{11}{2}$ e) $\frac{13}{2}$

e since $g'(x) = 1/(2\sqrt{x})$ and by the product rule,

$(fg)'(4) = f'(4)g(4) + f(4)g'(4) = 3 \cdot 2 + 2 \cdot \left(\frac{1}{4}\right) = 6 + \frac{1}{2} = \frac{13}{2}$

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10. Evaluate: $\frac{d}{dx}\sec(2x) =$

a) $\tan^2(2x)$ b) $2\tan^2(2x)$ c) $\sec(2x)\tan(2x)$

d) $2\sec(2x)\tan(2x)$ e) $4\sec(2x)\tan(2x)$

d since by the chain rule we have $\frac{d}{dx}\sec(2x) = 2\sec(2x)\tan(2x)$

11. Differentiating $x/y = \sin(x-y)$ implicitly with respect to x gives:

a) $\frac{y-xy'}{y^2} = (1-y')\cos(x-y)$ b) $\frac{y-xy'}{y^2} = y'\cos(x-y)$

c) $\frac{1}{y'} = (1-y')\cos(x-y)$ d) $\frac{y-xy'}{y^2} = \cos(x-y)$ e) $\frac{1}{y'} = \cos(x-y)$

a by using the quotient rule on the left and the chain rule on the right

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12. A spherical balloon is being inflated. Find the rate of increase of the surface area ($S = 4\pi r^2$) with respect to time when the radius r is 3cm if $\frac{dr}{dt} = 2$cm/s.

a) 6πcm^2/s b) 8πcm^2/s c) 24πcm^2/s d) 48πcm^2/s e) 72πcm^2/s

d since by the chain rule we have the rate of increase with respect to time is $\frac{dS}{dt} = 8\pi r\frac{dr}{dt}$ and so for $r = 3$ and $\frac{dr}{dt} = 2$, $\frac{dS}{dt} = 48\pi$. The units are as given.

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13. The linearization of $\ln(x)$ at $a = 1$ is

a) $L(x) = x$ b) $L(x) = \frac{1}{x}(x-1)$ c) $L(x) = x - 1$ d) $L(x) = \frac{1}{x} - 1$ e) undefined

c since, in general, linearization is given by $L(x) = f(a) + f'(a)(x-a)$. Here $f(x) = \ln(x)$, so $f(a) = f(1) = \ln(1) = 0$ and $f'(a) = \frac{1}{a} = 1$. This gives us $L(x) = x - 1$.

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14. On the interval $[-1,2]$ the absolute maximum of $h(x) = x^2$ is:

a) -1 b) 0 c) 1 d) 2 e) 4

e since the absolute maximum will occur either at a critical number (0, since $h'(x) = 2x = 0$ only for $x = 0$) or at one of the endpoints. Checking at $-1,0,2$, the maximum value is $h(2) = 4$.

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15. For $f(x) = e^{-2x}$ on the interval $[0,2]$ the value of c that satisfies the conclusion of the Mean Value Theorem is closest to

a) 0 b) 0.7 c) 1.0 d) 1.3 e) 2.0

b since we're looking for c such that $f'(c) = \frac{f(b)-f(a)}{b-a}$, in other words

$-2e^{-2c} = \frac{e^{-4}-e^0}{2-0} \cong -0.49084$, so $e^{-2c} \cong 0.24542$, and taking logs, $-2c \cong -1.4048$, so $c \cong 0.7$.

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16. Suppose that f is continuous everywhere and that $f'(x) = 0$ for $x = 0, 3,$ and 5. Further, suppose $f'(x) > 0$ for $x < 0$, $3 < x < 5$, and $5 < x$. Also, $f'(x) < 0$ for $0 < x < 3$. Then f has a local minimum at $x =$

a) 0 b) 3 c) 4 d) 5 e) not enough information

b since the local minimum will occur at the critical number 3 because f decreases on the left of 3 and increases on the right of 3.

17. Suppose $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} f'(x) = 3$ and $g(x) = \sin(2x)$. Then $\lim_{x\to 0}\frac{f(x)}{g(x)} =$

a) 0 b) $1/2$ c) 1 d) $3/2$ e) 3

d since $\lim_{x\to 0} g(x) = 0$ we can apply L'Hospital's rule, and since $g'(x) = 2\cos(2x)$ and so $\lim_{x\to 0} g'(x) = 2$ we have $\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)} = \frac{3}{2}$

18. Find the most general expression for function h if the second derivative is given by $h''(x) = e^x$:

a) $h(x) = e^x$ b) $h(x) = -e^{-x}$ c) $h(x) = Ce^x + D$

d) $h(x) = e^x + Cx + C$ e) $h(x) = e^x + Cx + D$

e since $h'(x) = e^x + C$ and $h(x) = e^x + Cx + D$. The two added constants need not be the same.

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19. The velocity of a plane traveling in a straight line is given at 12 second intervals:

t (s)	0	12	24	36	48	60
v (m/s)	100	110	115	115	120	125

Estimating the distance traveled during this minute of time using the velocities at the beginning of the time intervals gives:

a) 6720 m b) 6795 m c) 6870 m d) 6945 m e) 7020 m

a since $\Delta t = 12$, $dist \cong 100\Delta t + 110\Delta t + 115\Delta t + 115\Delta t + 120\Delta t = 560\Delta t = 6720$. Correct units of distance here are meters.

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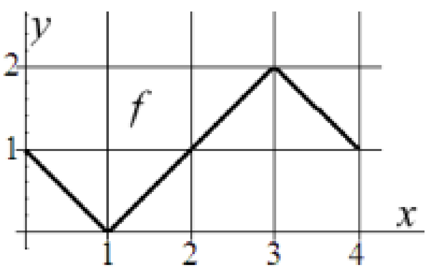
20. For f as sketched, and g defined by

$$g(x) = \int_0^x f(t)dt$$

what is $g(3)$?

a) 2 b) 2.5 c) 3 d) 3.5 e) 4

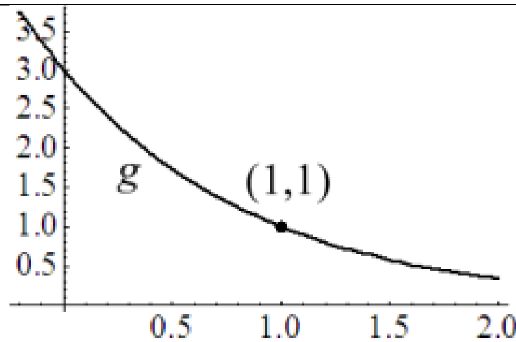


b since $g(3) = \int_0^3 f(t)dt = \int_0^1 f(t)dt + \int_1^3 f(t)dt$ equals the sum of the area of two triangles, with height 1 and base 1 (area=0.5) and with height 2 and base 2 (area=2) for total area 2.5

Part Two. Longer Answers (50%). These are not multiple choice. Again, SHOW ALL YOUR WORK ON THESE TEST PAGES. **CALCULATORS PERMITTED FOR ALL PROBLEMS EXCEPT 27, 28, 29, 30.**

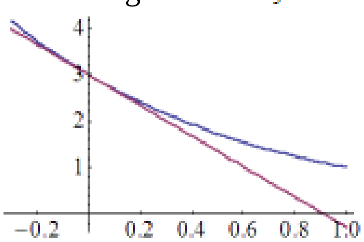
21. a) Find the exponential function $g(x) = Ca^x$ with the given graph.

b) Give an equation for the tangent line to the graph of g at $x = 0$.



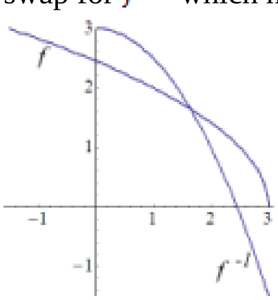
a) Since from the graph $g(0) = 3$ and from the equation $g(0) = Ca^0 = C$ we have $C = 3$. And since from the graph $g(1) = 1$ and from the equation $g(1) = Ca^1 = 3a$, $a = 1/3$. So $g(x) = 3\left(\frac{1}{3}\right)^x$.

b) To differentiate $y = 3\left(\frac{1}{3}\right)^x$ we first take logs to get $\ln(y) = \ln(3) + x\ln(\frac{1}{3})$. Differentiating gives $\frac{1}{y}y' = \ln(\frac{1}{3})$ so $y' = y\ln(\frac{1}{3})$. (Or, recall the formula, $\frac{d}{dx}(a^x) = a^x\ln(a)$.) At the point $(0,3)$ the graph has a tangent line with slope $3\ln\left(\frac{1}{3}\right) = -3\ln(3)$. So the slope intercept equation of the tangent line is $y = -3\ln(3)x + 3$. (or, less accurately, $y = -3.2958x + 3$). Picture:



22. Find a formula for $f^{-1}(x)$, the inverse of the function given by $f(x) = \sqrt{6 - 2x}$. What is the domain of f^{-1}?

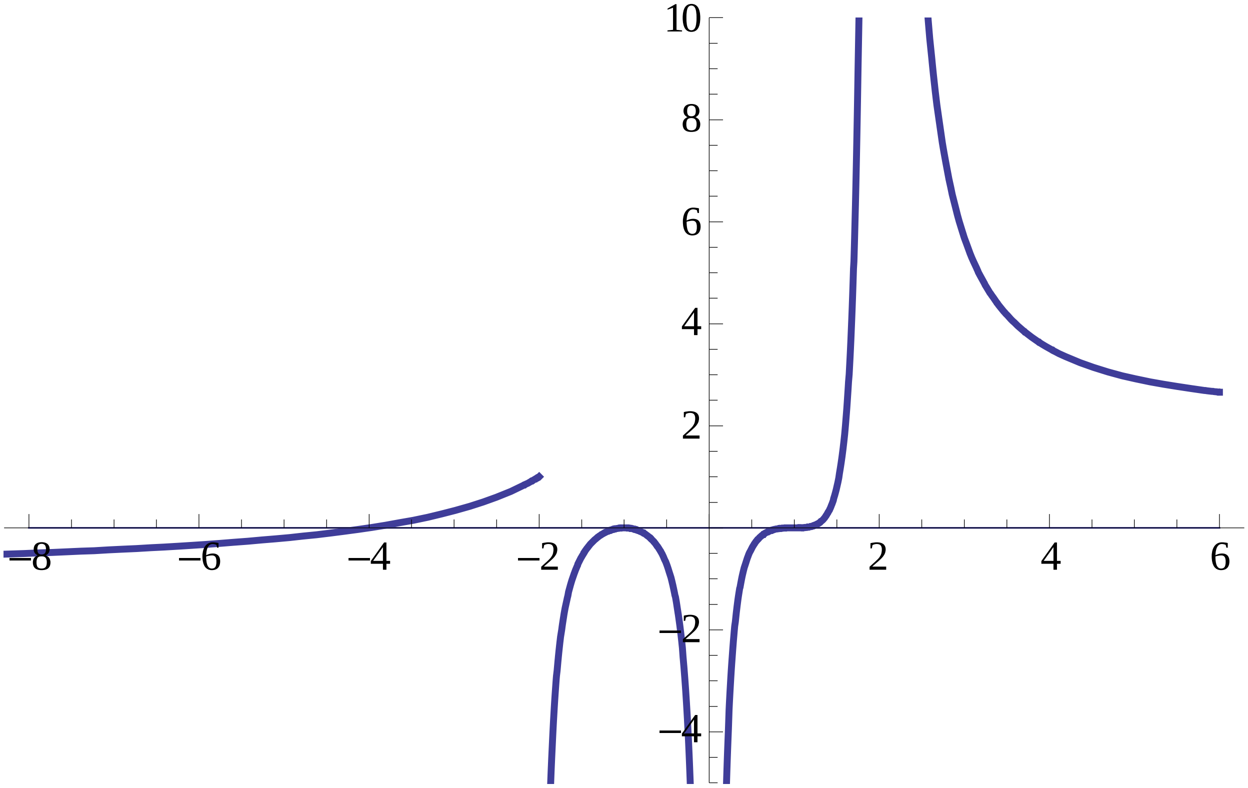
Solving $y = \sqrt{6 - 2x}$ for x we have $y^2 = 6 - 2x$ and so $2x = 6 - y^2$ and $x = 3 - y^2/2$. So the formula is $f^{-1}(x) = 3 - x^2/2$. Since the domain of f is $(-\infty, 3]$ and range is $[0, \infty)$, these swap for f^{-1} which has domain $[0, \infty)$. Both are sketched below:



23. Sketch the graph of a single function h that satisfies:

$\lim_{x\to-\infty} h(x) = -1$, $\lim_{x\to-2^-} h(x) = 1$, $\lim_{x\to-2^+} h(x) = -\infty$,

$\lim_{x\to-1} h(x) = 0$, $\lim_{x\to 0} h(x) = -\infty$, $\lim_{x\to 2} h(x) = \infty$, $\lim_{x\to\infty} h(x) = 2$

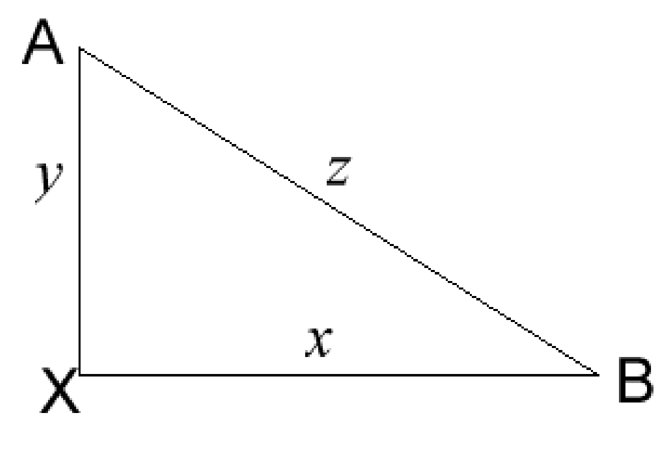


24. Showing all steps, use logarithmic differentiation to find $\frac{dy}{dx}$ for
$y = (\tan(x))^{1/x}$

Taking the logarithm we get $\ln(y) = \frac{1}{x}\ln(\tan(x))$, so differentiating using the chain and product rules : $\frac{1}{y}y' = -\frac{1}{x^2}\ln(\tan(x)) + \frac{1}{x}\frac{\sec^2(x)}{\tan(x)}$ which can be rewritten as

$$\frac{dy}{dx} = \frac{1}{x^2}\left(\frac{x}{\sin(x)\cos(x)} - \ln(\tan(x))\right)(\tan(x))^{1/x}$$

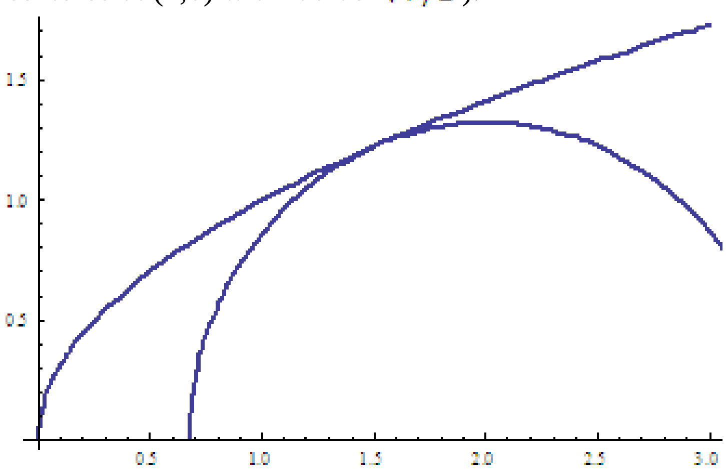
25. At noon ship A is 60 km north of location X and sailing north at 10 km/hr. At the same time ship B is 80 km east of X and sailing WEST at 13 km/hr. At what rate is the distance between the ships changing?



Using the figure, we have $\frac{dy}{dt} = 10$, $\frac{dx}{dt} = -13$, and $z^2 = x^2 + y^2$. So differentiating with respect to t, $2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$. At noon, $x = 60$ and $y = 80$, so $z = \sqrt{60^2 + 80^2} = 100$. Dividing by 2 and substituting, we get $100\frac{dz}{dt} = 60(10) + 80(-13) = -440$ so $\frac{dz}{dt} = -4.4$ and they are getting closer at a rate of 4.4 km/hr.

26. Find the point on the curve with equation $y = \sqrt{x}$ that is closest to the point $(2,0)$. [*Hint.* You might try minimizing d^2, the square of the distance from a point of the curve to $(2,0)$.]

Using the distance formula, for a point (x, y) on the curve $d^2 = (x-2)^2 + (y-0)^2 = (x^2 - 4x + 4) + x = x^2 - 3x + 4$. So to minimize we set the derivative to 0: $2x - 3 = 0$, giving us $x = 3/2$ and so $y = \sqrt{3/2}$. Since the second derivative is 2 which is positive, the point $(3/2, \sqrt{3/2})$ is where a global minimum occurs and so is the closest point. (The minimal distance is $\sqrt{7/2}$. Below is a sketch of the curve and the circle centered at (2,0) with radius $\sqrt{7/2}$).



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CALCULATORS NOT PERMITTED FOR 27, 28, 29, 30.

27. Use the Chain Rule and the Product Rule to give a proof of the Quotient Rule.
[*Hint:* Write $\frac{f(x)}{g(x)} = f(x)[g(x)]^{-1}$.]

$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) =$ by the definition of negative exponents

$\frac{d}{dx}\left(f(x)[g(x)]^{-1}\right) =$ by the Product Rule

$f'(x)[g(x)]^{-1} + f(x)\frac{d}{dx}\left([g(x)]^{-1}\right) =$ by the Chain Rule

$f'(x)[g(x)]^{-1} + f(x)\left(-[g(x)]^{-2}g'(x)\right) =$ by algebra

$\frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{(g(x))^2} =$ by algebra (getting common denominator)

$\frac{g(x)f'(x)}{(g(x))^2} - \frac{f(x)g'(x)}{(g(x))^2} =$ by subtraction

$\frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

28. Find the derivative $r'(t)$ if $r(t) = \ln(t) + \tan^{-1}(2t) + 3t\sin(t) + \cos(3t) - 2e^{-t}$.

$$r'(t) = \frac{1}{t} + \frac{2}{1 + 4t^2} + 3\sin(t) + 3t\cos(t) - 3\sin(3t) + 2e^{-t}$$

29. Sketch the graph of $y = x^4 - 4x^3$ and give:

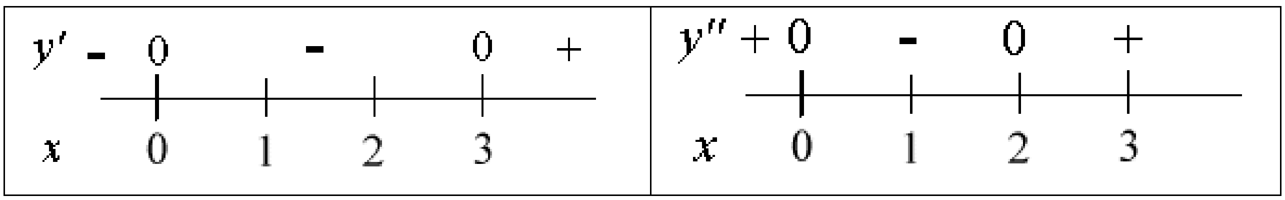
a) Open intervals of increase

b) Local minimum and local maximum values

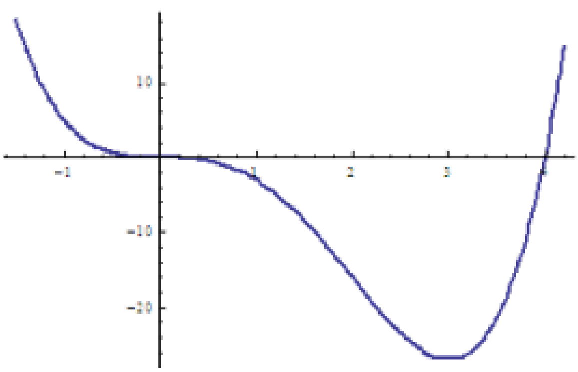
c) Open intervals of upward concavity

d) Points of inflection

We have $y' = 4x^3 - 12x^2 = 4x^2(x-3)$ and $y'' = 12x^2 - 24x = 12x(x-2)$. These give us the following sign charts for y' and y'':



So we have:
a) Open intervals of increase: where $f' > 0$: $(3, \infty)$
b) Local minimum and local maximum values: min at 3 of $3^4 - 4 \cdot 3^3 = -27$, no local max
c) Open intervals of upward concavity: where $f'' > 0$: $(-\infty, 0)$ and $(2, \infty)$
d) Points of inflection: where concavity changes: $(0,0)$ and $(2, -16)$
Also, note that $x^4 - 4x^3 = x^3(x-4)$ so the *x* intercepts are at 0 and 4.



30. Find the general indefinite integral: $\int (x^2 + x^{-2} + \sin(x))\,dx$

$\frac{1}{3}x^3 - x^{-1} - \cos(x) + C$