1. What is the DOMAIN of the function given by: \( h(x) = \frac{1}{\sqrt{4 - x^2}} \)
   a) \( x \neq 2 \)  b) all reals  c) \([-2, 2]\)  d) \((-2, 2]\)  e) \((0, \infty)\)
d since can’t take square root of negative and can’t divide by 0

2. Which of the following equations has the graph of an EVEN function?
   a) \( y = \sin(x) \)  b) \( y = \cos(x) \)  c) \( y = e^x \)  d) \( y = \ln(x) \)  e) \( y = 1/x \)
b since \( \cos(-x) = \cos(x) \) (also, graph is symmetric across the \( y \)-axis).

3. For \( f \) the function graphed and \( g(x) = e^x \), find the value of the composition at 1, namely, \( (f \circ g)(1) = f(g(1)) = \)
   a) 0  b) 1  c) 2  d) 3  e) \( e^x \)
d since \( e \approx 2.7 \), \( g(1) = e \) is between 2 and 3 and from the graph \( (f \circ g)(1) = f(e) = 3 \).

4. If \( \lim_{x \to a} f(x) = 3 \) and \( \lim_{x \to a} g(x) = 0 \) and \( \lim_{x \to a} h(x) = 1 \), then
   \( \lim_{x \to a} (f(x)g(x) + h(x)) = \)
   a) 0  b) 1  c) 2  d) 3  e) not enough information to evaluate
   since
   \begin{align*}
   \lim_{x \to a} (f(x)g(x) + h(x)) &= \lim_{x \to a} (f(x)) \lim_{x \to a} (g(x)) + \lim_{x \to a} (h(x)) \\
   &= 3 \cdot 0 + 1 = 1
   \end{align*}
5. Based on the given graph of \( g \), the left-hand limit as \( x \) approaches 1 from the left hand side is given by
\[
\lim_{x \to 1^-} g(x) =
\]
\[
(\text{0) 1 2 3 e) does not exist}
\]
\[
\text{since for } x \text{ close to, but less than, 1, } g(x) \text{ is close to 3.}
\]

6. Which equation is of a horizontal asymptote to the graph of
\[
y = \frac{2x^2 + 1}{x^2 - 4x + 3}
\]
\[
a) x = 1 \quad b) x = 3 \quad c) y = 0 \quad d) y = 1 \quad e) y = 2
\]
\[
\text{since } \lim_{x \to \pm \infty} \frac{2x^2 + 1}{x^2 - 4x + 3} = \lim_{x \to \pm \infty} \frac{2 + \left(\frac{1}{x}\right)}{1 - 4\left(\frac{1}{x}\right) + 3\left(\frac{1}{x^2}\right)} = 2
\]

7. Suppose that the table shows the percentage \( P \) of the population in Europe that uses cell phones in each year.

<table>
<thead>
<tr>
<th>Year</th>
<th>2000</th>
<th>2001</th>
<th>2002</th>
<th>2003</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P )</td>
<td>52</td>
<td>61</td>
<td>73</td>
<td>82</td>
</tr>
</tbody>
</table>

What is the average rate of cell phone growth from 2000 to 2003? (Units are percent/yr.)
\[
a) 10 \quad b) 15 \quad c) 30 \quad d) 82 \quad e) none of these
\]
\[
\text{since average rate} = \frac{\text{change of } P}{\text{change of Year}} = \frac{82 - 52}{2003 - 2000} = 10
\]

8. Which is an equation of the tangent line to \( y = x^2 - 2x^3 \) at \((-1,3)\)?
\[
a) y - 3 = (2x - 6x^2)(x + 1) \quad b) y - 3 = -8(x + 1) \quad c) y = -8x \quad d) y + 1 = -8(x - 3) \quad e) none of these
\]
\[
\text{since } y' = 2x - 6x^2, \text{ so at } (-1,3) \text{ slope is given by } m = -2 - 6 = -8. \text{ Point-slope form of the line gives b.}
\]

9. If \( f(4) = 2 \) and \( f'(4) = 3 \) and \( g(x) = \sqrt{x} \), then the derivative of the product at 4 is \( (fg)'(4) = \)
\[
a) 4 \quad b) 6 \quad c) 8 \quad d) \frac{11}{2} \quad e) \frac{13}{2}
\]
\[
\text{since } g'(x) = 1/(2\sqrt{x}) \text{ and by the product rule,}
\]
\[
(fg)'(4) = f'(4)g(4) + f(4)g'(4) = 3 \cdot 2 + 2 \cdot \left(\frac{1}{4}\right) = 6 + \frac{1}{2} = \frac{13}{2}
\]

10. Evaluate: \( \frac{d}{dx} \sec(2x) = \)
\[
a) \tan^2(2x) \quad b) 2\tan^2(2x) \quad c) \sec(2x) \tan(2x) \quad d) 2 \sec(2x) \tan(2x) \quad e) 4 \sec(2x) \tan(2x)
\]
\[
\text{since by the chain rule we have } \frac{d}{dx} \sec(2x) = 2 \sec(2x) \tan(2x)
\]
11. Differentiating $x/y = \sin(x - y)$ implicitly with respect to $x$ gives:

a) $\frac{3y-xy'}{y^2} = (1-y')\cos(x-y)$

b) $\frac{y-xy'}{y^2} = y'\cos(x-y)$

c) $\frac{1}{y'} = (1-y')\cos(x-y)$

d) $\frac{y-xy'}{y^2} = \cos(x-y)$

e) $\frac{1}{y'} = \cos(x-y)$

b) by using the quotient rule on the left and the chain rule on the right

12. A spherical balloon is being inflated. Find the rate of increase of the surface area ($S = 4\pi r^2$) with respect to time when the radius $r$ is 3cm if $\frac{dr}{dt} = 2$cm/s.

a) $6\pi$cm$^2$/s b) $8\pi$cm$^2$/s c) $24\pi$cm$^2$/s d) $48\pi$cm$^2$/s e) $72\pi$cm$^2$/s

a) since by the chain rule we have the rate of increase with respect to time is $\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$ and so for $r = 3$ and $\frac{dr}{dt} = 2$, $\frac{dS}{dt} = 48\pi$. The units are as given.

13. The linearization of $\ln(x)$ at $a = 1$ is

L(x) = x  b) $L(x) = \frac{1}{x}(x-1)$  c) $L(x) = x - 1$  d) $L(x) = \frac{1}{x} - 1$  e) undefined

c) since, in general, linearization is given by $L(x) = f(a) + f'(a)(x-a)$. Here $f(x) = \ln(x)$, so $f(a) = f(1) = \ln(1) = 0$ and $f'(a) = \frac{1}{a} = 1$. This gives us $L(x) = x - 1$.

14. On the interval $[-1,2]$ the absolute maximum of $h(x) = x^2$ is:

a) $-1$  b) $0$  c) $1$  d) $2$  e) $4$

e) since the absolute maximum will occur either at a critical number ($0$, since $h'(x) = 2x = 0$ only for $x = 0$) or at one of the endpoints. Checking at $-1,0,2$, the maximum value is $h(2) = 4$.

15. For $f(x) = e^{-2x}$ on the interval $[0,2]$ the value of $c$ that satisfies the conclusion of the Mean Value Theorem is closest to

a) $0$  b) $0.7$  c) $1.0$  d) $1.3$  e) $2.0$

e) since we’re looking for $c$ such that $f'(c) = \frac{f(b) - f(a)}{b-a}$, in other words

$-2e^{-2c} = \frac{e^{-4} - e^{0}}{2-0} \approx -0.49084$, so $e^{-2c} \approx 0.24542$, and taking logs, $-2c \approx -1.40458$, so $c \approx 0.7$.

16. Suppose that $f$ is continuous everywhere and that $f'(x) = 0$ for $x = 0,3$, and 5. Further, suppose $f'(x) > 0$ for $x < 0,3 < x < 5$, and $5 < x$. Also, $f'(x) < 0$ for $0 < x < 3$. Then $f$ has a local minimum at $x =$

a) $0$  b) $3$  c) $4$  d) $5$  e) not enough information

e) since the local minimum will occur at the critical number 3 because $f$ decreases on the left of 3 and increases on the right of 3.
17. Suppose \( \lim_{x \to 0} f(x) = 0 \) and \( \lim_{x \to 0} f'(x) = 3 \) and \( g(x) = \sin(2x) \). Then \( \lim_{x \to 0} \frac{f(x)}{g(x)} = \)
a) 0 b) \( \frac{1}{2} \) c) 1 d) \( \frac{3}{2} \) e) 3

Since \( \lim_{x \to 0} g(x) = 0 \) we can apply L’Hospital’s rule, and since \( g'(x) = 2\cos(2x) \) and so \( \lim_{x \to 0} g'(x) = 2 \) we have \( \lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \frac{3}{2} \)

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18. Find the most general expression for function \( h \) if the second derivative is given by \( h''(x) = e^x \):

a) \( h(x) = e^x \) b) \( h(x) = -e^{-x} \) c) \( h(x) = Ce^x + D \)
d) \( h(x) = e^x + Cx + C \) e) \( h(x) = e^x + Cx + D \)

Since \( h'(x) = e^x + C \) and \( h(x) = e^x + Cx + D \). The two added constants need not be the same.

19. The velocity of a plane traveling in a straight line is given at 12 second intervals:

<table>
<thead>
<tr>
<th>( t ) (s)</th>
<th>0</th>
<th>12</th>
<th>24</th>
<th>36</th>
<th>48</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v ) (m/s)</td>
<td>100</td>
<td>110</td>
<td>115</td>
<td>115</td>
<td>120</td>
<td>125</td>
</tr>
</tbody>
</table>

Estimating the distance traveled during this minute of time using the velocities at the beginning of the time intervals gives:

a) 6720 m b) 6795 m c) 6870 m d) 6945 m e) 7020 m

Since \( \Delta t = 12, \text{dist} = 100\Delta t + 110\Delta t + 115\Delta t + 115\Delta t + 120\Delta t = 560\Delta t = 6720 \).

Correct units of distance here are meters.

20. For \( f \) as sketched, and \( g \) defined by \( g(x) = \int_0^x f(t) \, dt \)

what is \( g(3) \)?

a) 2 b) 2.5 c) 3 d) 3.5 e) 4

Since \( g(3) = \int_0^3 f(t) \, dt = \int_0^1 f(t) \, dt + \int_1^3 f(t) \, dt \) equals the sum of the area of two triangles, with height 1 and base 1 (area=0.5) and with height 2 and base 2 (area=2) for total area 2.5
21. a) Find the exponential function 
\[ g(x) = C \alpha^x \] with the given graph.

b) Give an equation for the tangent line to the graph of \( g \) at \( x = 0 \).

\[ g(0) = 3 \text{ and from the equation } g(0) = C \alpha^0 = C \text{ we have } C = 3. \]

\[ g(1) = 1 \text{ and from the equation } g(1) = C \alpha^1 = 3\alpha, \alpha = 1/3. \text{ So } g(x) = 3 \left( \frac{1}{3} \right)^x. \]

b) To differentiate \( y = 3 \left( \frac{1}{3} \right)^x \) we first take logs to get \( \ln(y) = \ln(3) + x \ln\left( \frac{1}{3} \right) \). Differentiating gives \( \frac{1}{y} y' = \ln\left( \frac{1}{3} \right) \) so \( y' = y \ln\left( \frac{1}{3} \right) \).

At the point \((0,3)\) the graph has a tangent line with slope \( 3 \ln\left( \frac{1}{3} \right) = -3 \ln(3) \). So the slope intercept equation of the tangent line is \( y = -3 \ln(3)x + 3 \). (Or, less accurately, \( y = -3.2958x + 3 \)). Picture:

22. Find a formula for \( f^{-1}(x) \), the inverse of the function given by \( f(x) = \sqrt{6 - 2x} \). What is the domain of \( f^{-1} \)?

Solving \( y = \sqrt{6 - 2x} \) for \( x \) we have \( y^2 = 6 - 2x \) and so \( 2x = 6 - y^2 \) and \( x = 3 - y^2/2 \). So the formula is \( f^{-1}(x) = 3 - x^2/2 \). Since the domain of \( f \) is \((-\infty, 3]\) and range is \([0, \infty)\), these swap for \( f^{-1} \) which has domain \([0, \infty)\). Both are sketched below:
23. Sketch the graph of a single function \( h \) that satisfies:
\[
\lim_{x \to -\infty} h(x) = -1, \quad \lim_{x \to -2} h(x) = 1, \quad \lim_{x \to 2^+} h(x) = -\infty, \\
\lim_{x \to -1} h(x) = 0, \quad \lim_{x \to 0} h(x) = -\infty, \quad \lim_{x \to 2} h(x) = \infty, \quad \lim_{x \to \infty} h(x) = 2
\]

24. Showing all steps, use logarithmic differentiation to find \( \frac{dy}{dx} \) for
\[
y = (\tan(x))^{1/x}
\]

Taking the logarithm we get \( \ln(y) = \frac{1}{x} \ln(\tan(x)) \), so differentiating using the chain and product rules:
\[
\frac{1}{y} y' = -\frac{1}{x^2} \ln(\tan(x)) + \frac{1}{x} \frac{\sec^2(x)}{\tan(x)}
\]
which can be rewritten as
\[
\frac{dy}{dx} = \frac{1}{x^2} \left( \frac{x}{\sin(x) \cos(x)} - \ln(\tan(x)) \right) (\tan(x))^{1/x}
\]
25. At noon ship A is 60 km north of location X and sailing north at 10 km/hr. At the same time ship B is 80 km east of X and sailing WEST at 13 km/hr. At what rate is the distance between the ships changing?

Using the figure, we have \( \frac{dy}{dt} = 10 \), \( \frac{dx}{dt} = -13 \), and \( z^2 = x^2 + y^2 \). So differentiating with respect to \( t \),

\[
2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}.
\]

At noon, \( x = 60 \) and \( y = 80 \), so \( z = \sqrt{60^2 + 80^2} = 100 \).

Dividing by 2 and substituting, we get

\[
100 \frac{dz}{dt} = 60(10) + 80(-13) = -440
\]

so \( \frac{dz}{dt} = -4.4 \) and they are getting closer at a rate of 4.4 km/hr.

26. Find the point on the curve with equation \( y = \sqrt{x} \) that is closest to the point \((2,0)\). [Hint. You might try minimizing \( d^2 \), the square of the distance from a point of the curve to \((2,0)\).]

Using the distance formula, for a point \((x, y)\) on the curve

\[
d^2 = (x - 2)^2 + (y - 0)^2 = (x^2 - 4x + 4) + x = x^2 - 3x + 4.
\]

So to minimize we set the derivative to 0: \( 2x - 3 = 0 \), giving us \( x = 3/2 \) and so \( y = \sqrt{3/2} \). Since the second derivative is 2 which is positive, the point \((3/2, \sqrt{3/2})\) is where a global minimum occurs and so is the closest point. (The minimal distance is \( \sqrt{7/2} \). Below is a sketch of the curve and the circle centered at \((2,0)\) with radius \( \sqrt{7/2} \).)
Page 8 is BLANK and can be used as SCRATCH PAPER
27. Use the Chain Rule and the Product Rule to give a proof of the Quotient Rule. [Hint: Write \( \frac{f(x)}{g(x)} = f(x)[g(x)]^{-1} \).]

\[
\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = f'(x)[g(x)]^{-1} - f(x)[g(x)]^{-2}g'(x) = f'(x)[g(x)]^{-1} + f(x)\frac{d}{dx}\left( [g(x)]^{-1} \right)
\]

by the definition of negative exponents

by the Product Rule

by the Chain Rule

by algebra

by algebra (getting common denominator)

by subtraction

28. Find the derivative \( r'(t) \) if \( r(t) = \ln(t) + \tan^{-1}(2t) + 3t \sin(t) + \cos(3t) - 2e^{-t}. \)

\[
r'(t) = \frac{1}{t} + \frac{2}{1 + 4t^2} + 3 \sin(t) + 3t \cos(t) - 3 \sin(3t) + 2e^{-t}
\]
29. Sketch the graph of \( y = x^4 - 4x^3 \) and give:

a) Open intervals of increase

b) Local minimum and local maximum values

c) Open intervals of upward concavity

d) Points of inflection

We have \( y' = 4x^3 - 12x^2 = 4x^2(x-3) \) and \( y'' = 12x^2 - 24x = 12x(x-2) \). These give us the following sign charts for \( y' \) and \( y'' \):

\[
\begin{array}{cccc|c|c}
   x & 0 & 1 & 2 & 3 & - & 0 & + \\
   y' & - & - & - & 0 & + & & \\
   y'' & 0 & - & 0 & + & & \\
\end{array}
\]

So we have:

a) Open intervals of increase: where \( f' > 0 \): \((3, \infty)\)

b) Local minimum and local maximum values: min at 3 of \( 3^4 - 4 \cdot 3^3 = -27 \), no local max

c) Open intervals of upward concavity: where \( f'' > 0 \): \((-\infty, 0) \) and \((2, \infty)\)

d) Points of inflection: where concavity changes: \((0,0)\) and \((2, -16)\)

Also, note that \( x^4 - 4x^3 = x^3(x-4) \) so the \( x \) intercepts are at 0 and 4.

30. Find the general indefinite integral: \( \int (x^2 + x^{-2} + \sin(x)) \, dx \)

\[
\frac{1}{3}x^3 - x^{-1} - \cos(x) + C
\]