NAME: DARCHANGELO - KEY ALPHA NUMBER: ____________

INSTRUCTOR: ____________ SECTION: ____________

CALCULUS I (SM121, SM121A, SM131) FINAL EXAMINATION

1930-2230 Tuesday 14 December 2010 SHOW ALL WORK ON THIS TEST PACKAGE

Tear off the last sheet of this exam. It contains problems 27, 28, 29, 30. Do these problems first WITHOUT YOUR CALCULATOR. When you are finished with these 4 problems, return the sheet to your instructor, take out your calculator, and complete the rest of the exam.

PART ONE: MULTIPLE CHOICE (50%). The first 20 problems are multiple-choice. Fill in the best answer on your Scantron bubble sheet. Write your name, alpha number, instructor and section on this test and your bubble sheet and bubble in your alpha number. There is no extra penalty for wrong answers on the multiple choice. Show all your scratch work on this test.

CALCULATORS ARE PERMITTED FOR THIS MULTIPLE CHOICE SECTION.

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1. Find an equation for the line going through the point (2 , 2) and perpendicular to the line $y = 2x+1$.

$y - y_0 = m(x - x_0)$
$\Rightarrow y - 2 = -\frac{1}{2}(x-2)$
$\Rightarrow y = -\frac{1}{2}x + 1 + 2$
$\Rightarrow y = -\frac{1}{2}x + 3$

a) $y = \dfrac{-1}{2x+1}$ b) $y = \frac{1}{2}x+1$ c) $y = -\frac{1}{2}x+1$

d d) $y = -\frac{1}{2}x+3$ e) $y = -2x+6$

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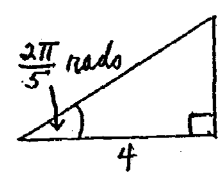
2. Find the side x for the triangle on the right. The triangle is not drawn to scale.



$\frac{x}{4} = \tan\left(\frac{2\pi}{5}\right)$
$\Rightarrow x = 4\tan\left(\frac{2\pi}{5}\right)$
$\doteq 12.3$

a a) 12.3 b) 10.2 c) 3.0 d) $\pi/3$ e) 8.1

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3. If $\frac{\pi}{2} < \theta < \pi$ and $\sin(\theta) = 0.6$, find $\cos(\theta)$.

$\cos^2(\theta) + \sin^2(\theta) = 1$
$\Rightarrow \cos(\theta) = \pm\sqrt{1 - \sin^2(\theta)}$
$= \pm\sqrt{1 - .36}$
$\left(\frac{\pi}{2} < \theta < \pi \Rightarrow \text{minus}\right) = -\sqrt{.64} = -0.8$

d a) 0.4 b) -0.4 c) 0.8 d) -0.8 e) $\pi/3$

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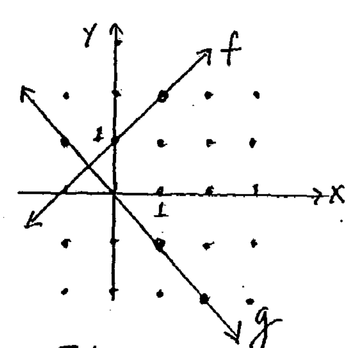
4. Find $g(f^{-1}(2))$ if f^{-1} is the inverse of the function f graphed on the right along with the function g.



$g(f^{-1}(2))$
$= g(1)$
$= -1$

a a) -1 b) 1 c) -2

d) 2 e) 0

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5. Find $\lim\limits_{x\to 0^+} \dfrac{f(x)}{g(x)}$ where f and g are the functions graphed in the previous problem 4.

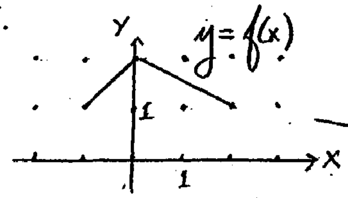
$\lim\limits_{x\to 0^+} \dfrac{f(x) \nearrow 1}{g(x) \searrow 0 \text{ (negative)}} = -\infty$

e a) 1 b) -1 c) 0 d) ∞ e) $-\infty$

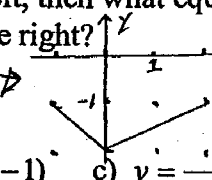
6. If a bacteria population starts with 100 bacteria and doubles every 5 hours, the number of bacteria after t hours is $n = 100(2^{t/5})$. In how many hours will the population reach 50,000 bacteria?

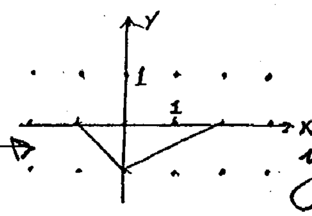
$100(2^{t/5}) = 50,000$
$2^{t/5} = 500$
$\ln(2^{t/5}) = \ln(500)$
$\frac{t}{5}\ln(2) = \ln(500)$
$t = 5 \cdot \ln(500)/\ln(2) \doteq 44.8$

b a) 12.3 b) 44.8 c) 3.0 d) 23.4 e) 8.1

7. 

If the equation $y = f(x)$ is graphed on the left, then what equation is graphed on the right?

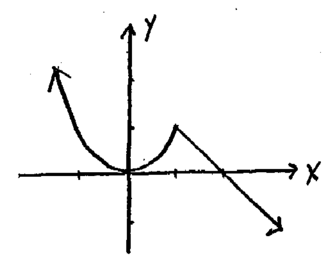




a a) $y = -f(x)+1$ b) $y = -2f(x-1)$ c) $y = \dfrac{1}{f(x)}$ d) $y = f^{-1}(x)$ e) $y = -2f(x+1)$

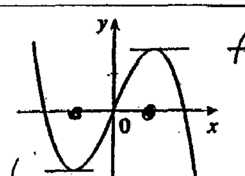
8. A function is defined piecewise by $f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ -x+2 & \text{if } x > 1 \end{cases}$.

Which of the following statements is true? (hint: sketch the graph.)

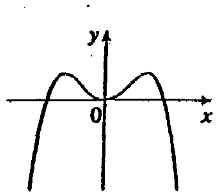


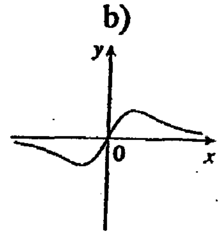
a a) f is continuous everywhere, but not differentiable everywhere.
b) f is differentiable everywhere, but not continuous everywhere.
c) f is continuous everywhere, but $\lim_{x \to 1} f(x)$ does not exist.
d) $\lim_{x \to 1^+} f(x) \ne \lim_{x \to 1^-} f(x)$.
e) none of the above statements is true.

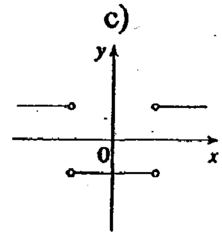
9. If f is a function whose graph is shown on the right, which of the following could be the graph for f' ?

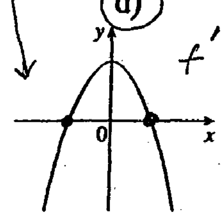


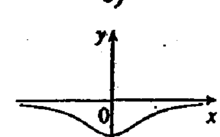
d

a)


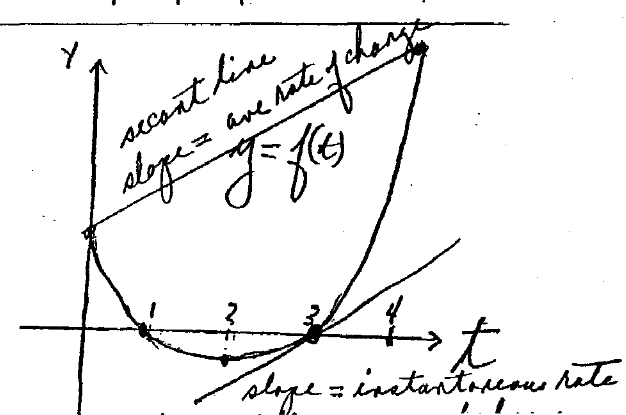
b)


c)


d)


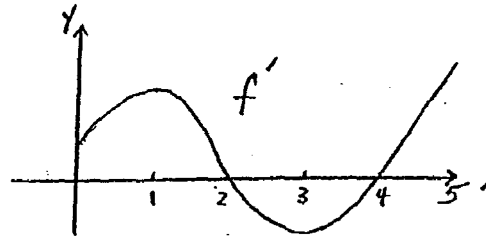
e)


10. Use the graph on the right to find a number t where the instantaneous rate of change of f is equal to the average rate of change of f over the interval [0 , 4].



d a) 0 b) 1 c) 2 d) 3 e) 4

11. If the graph of f' (*not* f) is shown on the right, then the graph of f is concave down on what interval?



$f''=(f')'<0$ where f' is decreasing.

c a) $(0, 2)$ b) $(2, 4)$ c) $(1, 3)$ d) $(4, 5)$ e) $(2, 5)$

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12. Use implicit differentiation to find $\frac{dy}{dx}$ at the point $(1, 1)$ for curve defined by $x^2 + xy + y^2 = 3$.

b a) 1 b) -1 c) 2 d) -2 e) 3

$\frac{d}{dx}[x^2+xy+y^2] = \frac{d}{dx}[3]$
$\Rightarrow 2x + 1y + xy' + 2yy' = 0$
$\Rightarrow (x+2y)y' = -2x - y$
$\Rightarrow y' = \frac{-2x-y}{x+2y}\Big|_{(1,1)} = \frac{-3}{3} = -1$

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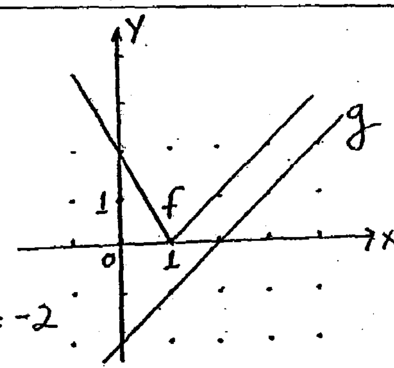
13. Use the chain rule and the graphs of the functions f and g on the right to find $\frac{d}{dx}[f(g(x))]$ at $x = 2$.



a) 0 b) -1 c) 2
d d) -2 e) 3

$[f(g(x))]' = f'(g(x))\cdot g'(x)\,\big|_{x=2}$
$= f'(g(2))\cdot g'(2) = f'(0)\cdot(1) = (-2)(1) = -2$

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14. Boyle's law for a compressed gas at a constant temperature states that $PV = C$ where C is a constant. At a certain instant, the volume V is 200 cm^3, the pressure P is 100 kPa, and the pressure is decreasing at the rate of $10\,kPa/\min$. At what rate ($cm^3/\min$) is the volume increasing at this instant?

a) 200 b) 10 c) 100
e d) 2000 e) 20

$\frac{d}{dt}[P(t)V(t)] = \frac{d}{dt}[C]$
$\Rightarrow P'V + PV' = 0$
$\Rightarrow (-10\,\frac{kPa}{min})(200\,cm^3) + (100\,kPa)(V') = 0$
$\Rightarrow V' = \frac{2000\,kPa\cdot cm^3/min}{100\,kPa} = 20\,cm^3/min$

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15. Find an equation for the line tangent to the parabola $y = x^2 + x$ at the point corresponding to $x = 1$.

b a) $y = 2x + 1$ b) $y = 3x - 1$ c) $y - 2 = (2x+1)(x-1)$
d) $y = \frac{x^3}{3} + \frac{x^2}{2} + 1$ e) $y = x + 1$

tangent line: $y - y_0 = m(x - x_0)$
$(x_0, y_0) = (1, 2)$
$m = y'\big|_{x=1} = 2x+1\big|_{x=1} = 3$
$\Rightarrow y - 2 = 3(x-1)$
$\Rightarrow y = 3x - 1$

16. If $f(x)=\dfrac{2}{x-1}$ and g is the inverse function of f then a formula for $g(x)$ is

b

a) $g(x)=\dfrac{2}{x+1}$ b) $g(x)=\dfrac{2+x}{x}$ c) $g(x)=\dfrac{2-x}{x}$

d) $g(x)=2\ln(x-1)$ e) $g(x)=\dfrac{2-x}{2+x}$

$y=\frac{2}{x-1} \Rightarrow (x-1)y=2$
$\Rightarrow xy-y=2$
$\Rightarrow x=\frac{2+y}{y}$
$\Rightarrow g^{-1}(x)=\frac{2+x}{x}$

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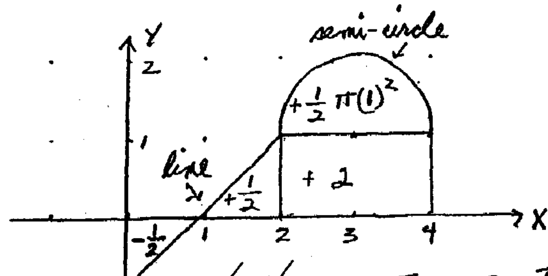
17. Determine $\int_0^4 f(x)dx$ for the function f whose graph on the right consists of a line segment and a semi-circle of radius one.



$-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}+2+\frac{\pi}{2}=2+\frac{\pi}{2}$

a

a) $2+\dfrac{\pi}{2}$ b) $2+\pi$ c) 4

d) $3+\dfrac{\pi}{2}$ e) $\dfrac{\pi^2}{2}$

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18. Approximate $\int_0^4 f(x)dx$ for the function f whose graph appears in problem 17 by dividing the interval [0 , 4] into n = 2 subintervals and using the midpoint rule (M_2).

$M_2 = f(1)\Delta x + f(3)\Delta x$
$= 0\cdot 2 + 2\cdot 2$
$= 4$

c

a) $2+\dfrac{\pi}{2}$ b) $2+\pi$ c) 4 d) $3+\dfrac{\pi}{2}$ e) $\dfrac{\pi^2}{2}$

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19. If a particle moves in a straight line with acceleration $a(t)=3t^2+e^t$, find its position $s(t)$ if initially its velocity is $v(0)=0$ and its position is $s(0)=0$.

$a(t)=3t^2+e^t$
$\Rightarrow v(t)=t^3+e^t+C_1$
$v(0)=0 \Rightarrow 0^3+\underbrace{e^0}_{1}+C_1=0 \Rightarrow C_1=-1$
$\Rightarrow v(t)=t^3+e^t-1$
$\Rightarrow s(t)=t^4/4+e^t-t+C_2$
$s(0)=0 \Rightarrow 0+e^0-0+C_2=0 \Rightarrow C_2=-1$
$\Rightarrow s(t)=t^4/4+e^t-t-1$

c

a) $s(t)=t^4/4+e^t$ b) $s(t)=t^4/4+e^t-1$

c) $s(t)=t^4/4+e^t-t-1$ d) $s(t)=16t^2+e^t$

e) $s(t)=t^3+e^t-1$

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20. The velocity of a swimmer during the first nine seconds of a race is given on the right at three second intervals. Estimate the distance traveled (in feet) during the first nine seconds by using the velocities at the right end of each time interval.

t (s)	0	3	6	9
v (ft/s)	0	4	6	7

dist $\doteq v(3)\Delta t+v(6)\Delta t+v(9)\Delta t$
$= 4\cdot 3+6\cdot 3+7\cdot 3$
$= (4+6+7)\,\text{ft/s}\cdot(3)\text{s}$
$= (17)\,\text{ft/s}\cdot(3)\text{s}$
$= 51\ \text{ft}$

c

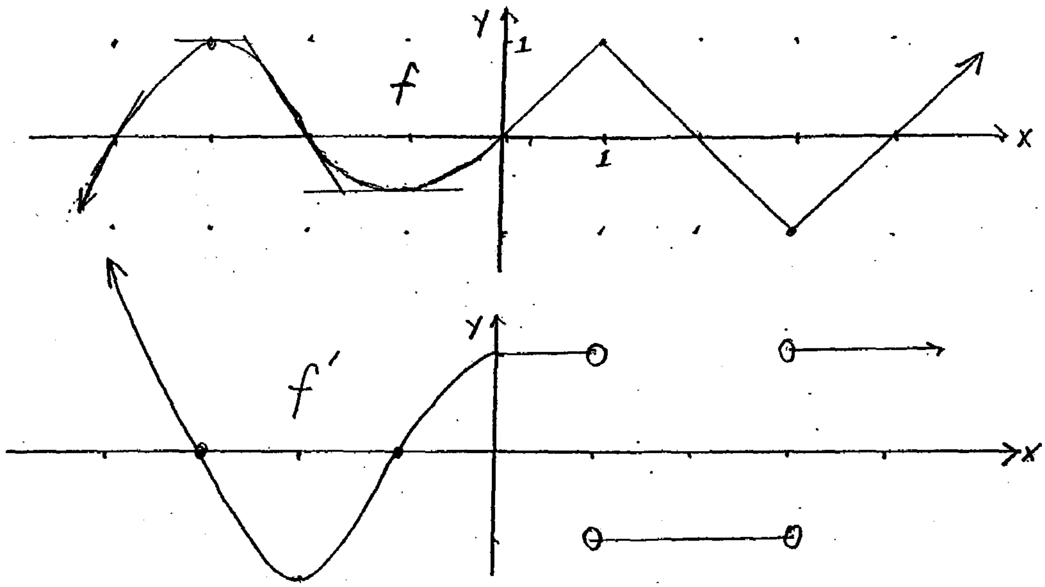
a) 10 b) 17 c) 51

d) 18 e) 7/9

NAME: ______________________ ALPHA NUMBER: __________
INSTRUCTOR: ______________________ SECTION: __________
CALCULUS I FINAL EXAMINATION 14 DEC 2010

Part Two. Longer Answers (50%). These are not multiple choice. Again, SHOW ALL YOUR WORK ON THESE TEST PAGES. **CALCULATORS PERMITTED FOR ALL PROBLEMS <u>EXCEPT</u> 27, 28, 29, 30.**

21. The graph of a function f is shown. Sketch the graph of its derivative f' on the axes below.

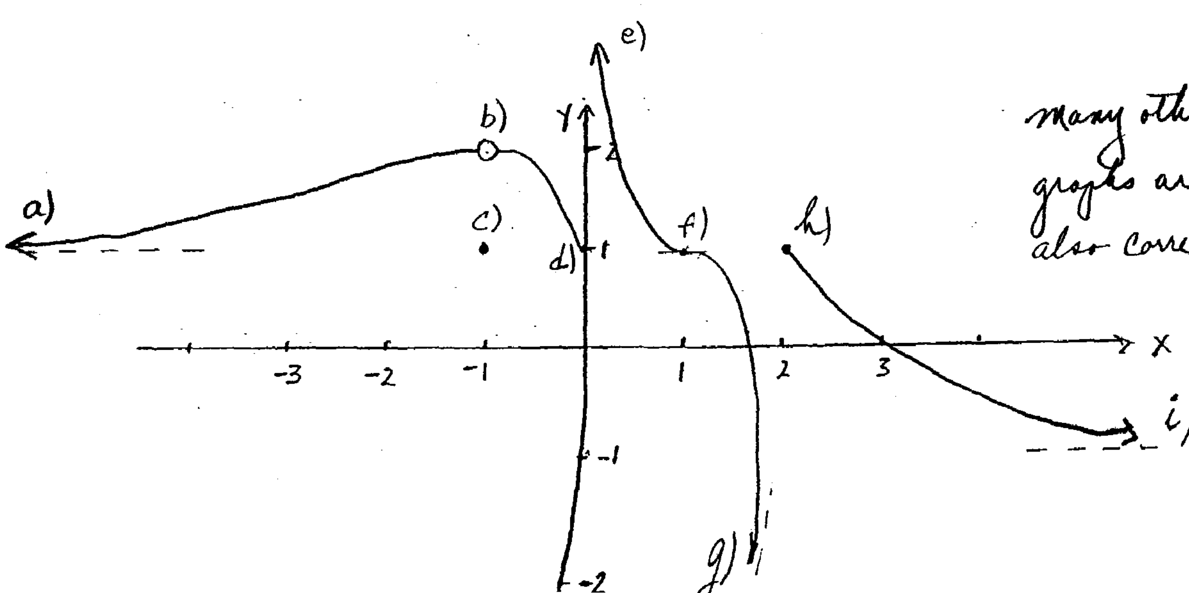


22. Sketch the graph <u>of a single function</u> (on the axes below) that satisfies all of the following:

a) $\lim_{x\to-\infty} f(x)=1$; b) $\lim_{x\to-1} f(x)=2$; c) $f(-1)=1$;

d) $\lim_{x\to0^-} f(x)=1$; e) $\lim_{x\to0^+} f(x)=\infty$; f) $f'(1)=0$;

g) $\lim_{x\to2^-} f(x)=-\infty$; h) $\lim_{x\to2^+} f(x)=1$; i) $\lim_{x\to\infty} f(x)=-1$.



Many other graphs are also correct.

23. Use the definition $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ to prove that if $f(x) = x^3$, then $f'(x) = 3x^2$.

$$(x+h)\left[(x+h)^2\right] = (x+h)\left[x^2 + 2xh + h^2\right] = x^3 + 2x^2h + \boxed{xh^2} + x^2h + \boxed{2xh^2} + h^3$$

$$f'(x) = \lim_{h \to 0}\left[\frac{(x+h)^3 - x^3}{h}\right] = \lim_{h \to 0} \frac{(\cancel{x^3} + 3x^2h + 3xh^2 + h^3) - \cancel{x^3}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}\left[3x^2 + 3xh + h^2\right]}{\cancel{h}} = \lim_{h \to 0} 3x^2 + 3x\overset{0}{h} + \overset{0}{h^2} = 3x^2$$

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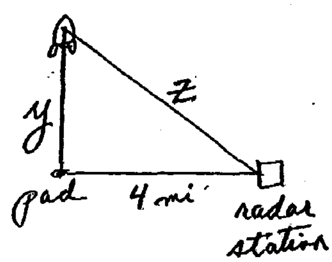
24. A rocket is launched vertically from a pad located 4 miles from a radar station. What is the altitude and speed of the rocket at the moment when the rocket is 5 miles from the radar station and its distance from the radar station is increasing at a rate of 300 mi/hr ?

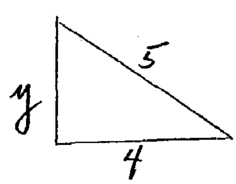


Find y (altitude) and $\frac{dy}{dt}$ (speed) when $z = 5$ mi and $dz/dt = 300$ mi/hr.

$$(z(t))^2 = (y(t))^2 + (4)^2$$

$$\Rightarrow \frac{d}{dt}\left[(z(t))^2\right] = \frac{d}{dt}\left[(y(t))^2 + 16\right]$$

$$\Rightarrow \cancel{2} z \frac{dz}{dt} = \cancel{2} y \frac{dy}{dt} + 0$$



$$\Rightarrow (5\text{ mi})\left(300\,\frac{\text{mi}}{\text{hr}}\right) = (3\text{ mi})\frac{dy}{dt}$$

$y^2 + 4^2 = 5^2$

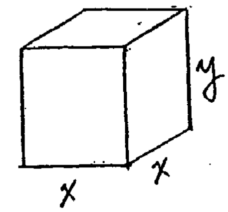
$y = \sqrt{25 - 16}$

$= 3$ mi

$$\Rightarrow \frac{dy}{dt} = \frac{1500}{3}\,\frac{\text{mi}}{\text{hr}} = 500 \text{ mi/hr (speed)}$$

when $y = 3$ mi (altitude)

25. A box with a square base and an open top must have a volume of 4 ft^3.

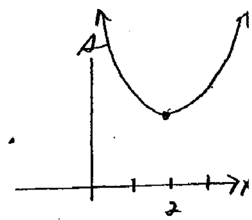


a) Find a formula for the total surface as a function of x alone.

surface area $A = x^2 + 4xy$ (bottom + 4 sides). $V = x \cdot x \cdot y = 4 \Rightarrow y = 4/x^2$

$\Rightarrow A = x^2 + 4x\left(\frac{4}{x^2}\right) \Rightarrow A(x) = x^2 + \frac{16}{x}$ where $x > 0$.

b) Find the dimensions of the box that minimize the total surface area.



$A'(x) = 2x - \frac{16}{x^2} = 0 \Rightarrow 2x = \frac{16}{x^2} \Rightarrow x^3 = 8 \Rightarrow x = 2.$

$A''(x) = 2 + \frac{32}{x^3} > 0$ for $x > 0 \Rightarrow$ concave up $\Rightarrow x = 2$ is a minimum

$\Rightarrow$ dimensions are $2 \times 2 \times 1$.

26. Suppose that the cost C of a day's worth of electricity at the Naval Academy is a function of the average outdoor temperature T. In other words, $C = f(T)$, where C is measured in dollars and T is measured in degrees Fahrenheit. Answer the following questions, making sure that you give the proper units.

a) What does $f(80) = 4000$ mean? If the average daily temp = 80 (degrees Fahrenheit) then a day's worth of electricity costs 4,000 (dollars).

b) If $f(80) = 4000$ and $f(85) = 5000$, what is the average rate of change of this function over the interval [80 , 85] ?

$\frac{f(85) - f(80)}{85 - 80} = \frac{5000 - 4000}{5} \frac{\text{dollars}}{\text{deg F}} = 200 \left(\text{dollars}/{}^\circ F\right)$

c) What does $f'(85) = 250$ mean? The instantaneous rate of change when the average daily temp $= 85°F$ is $250\left(\frac{\text{dollars}}{°F}\right)$.

d) If $f(85) = 5000$ and $f'(85) = 250$, find a linear approximation of the cost of a day's worth of electricity at the Naval Academy if the average outdoor temperature is 85.5 degrees Fahrenheit.

$f(85.5) = f(85) + f'(85)(85.5 - 85)$

$= \$5000 + \left(\frac{\$250}{°F}\right)(.5\,°F) = \$5000 + \$125 = 5{,}125$ dollars.

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CALCULATORS NOT PERMITTED FOR 27, 28, 29, 30.

27. Find $\dfrac{dy}{dx}$ for the following functions. (Do not simplify your answers.)

a) $y = e^{3x}\sin(5x) \Rightarrow y' = 3e^{3x}\sin(5x) + e^{3x}\cos(5x)\cdot 5$

b) $y = \dfrac{\sqrt{x}}{\ln(x)} = \dfrac{x^{1/2}}{\ln(x)} \Rightarrow y' = \dfrac{\frac{1}{2}x^{-\frac{1}{2}}\cdot\ln(x) - x^{\frac{1}{2}}\cdot\frac{1}{x}}{[\ln(x)]^2}$

c) $y = \sec(x) + \arctan(x) \Rightarrow y' = \sec(x)\tan(x) + \dfrac{1}{1+x^2}$

d) $y = x^{\cos(x)} \Rightarrow \ln(y) = \cos(x)\ln(x)$

$\Rightarrow \dfrac{1}{y}\dfrac{dy}{dx} = -\sin(x)\ln(x) + \cos(x)\cdot\dfrac{1}{x}$

$\Rightarrow \dfrac{dy}{dx} = x^{\cos(x)}\left[-\sin(x)\ln(x) + \dfrac{\cos(x)}{x}\right]$

28. Evaluate the following definite integral showing all of your steps:

a) $\int_{-2}^{1}(1-2x+3x^2)\,dx = x - x^2 + x^3 \Big|_{-2}^{1}$

$= (1-1+1) - (-2-(-2)^2+(-2)^3)$

$= (1) - (-2-4-8)$

$= (1) - (-14)$

$= 15$

b) $\int_0^{\pi/4}\sec^2(t)\,dt = \tan(t)\Big|_0^{\pi/4} = \tan(\pi/4) - \tan(0)$

$= 1 - 0$

$= 1$

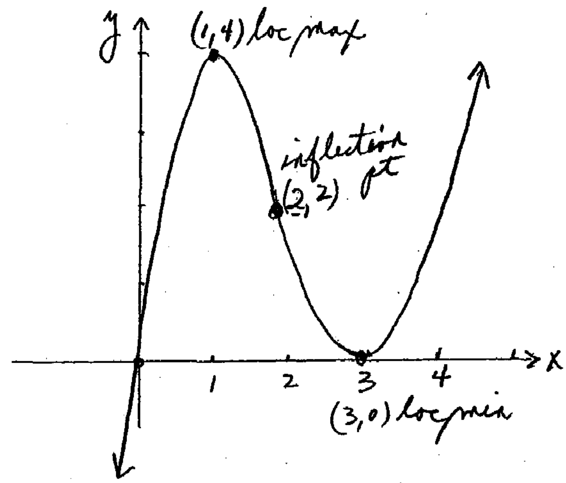
29. Use f' and f'' to graph $f(x) = x^3 - 6x^2 + 9x$. Label all relative maximums and minimums and inflection points.

$$f(x) = x^3 - 6x^2 + 9x = x(x^2 - 6x + 9)$$
$$= x(x-3)(x-3)$$
$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3)$$
$$= 3(x-3)(x-1)$$
$$f''(x) = 6x - 12 = 6(x-2)$$

− − 0 + + + 0 + + sign of f
(−1, 0, 1, 2, 3, 4)

+ + + 0 − − 0 + + sign of f'
(−1, 0, 1, 2, 3, 4)

− − − 0 + + sign of f''
(−1, 0, 1, 2, 3, 4)



30. Find the following limits. Show all work.

a) $\lim\limits_{x\to 0} \frac{x^2 + x - 12}{x - 3} = \frac{-12}{-3} = 4$ (can <u>not</u> use L'Hospital's Rule)

b) $\lim\limits_{x\to 3} \frac{(x^2 + x - 12)}{(x - 3)} \overset{L'H}{=} \lim\limits_{x\to 3} \frac{2x+1}{1} = 7$

or $\overset{alg}{=} \lim\limits_{x\to 3} \frac{(x+4)\cancel{(x-3)}}{\cancel{(x-3)}} = 7$

c) $\lim\limits_{x\to 0^+} x^2 \ln(x) \underset{alg}{=} \lim\limits_{x\to 0^+} \frac{\ln(x)}{1/x^2} \overset{L'H}{=} \lim\limits_{x\to 0^+} \frac{1/x}{-2/x^3} \overset{alg}{=} \lim\limits_{x\to 0^+} \frac{1}{x} \cdot \frac{x^3}{-2}$

$(0)(-\infty)$; $\ln(x) \to -\infty$; $1/x^2 \to +\infty$

$= \lim\limits_{x\to 0^+} -\frac{x^2}{2} = 0$