Part 1: Multiple Choice (50%)

Enter your answer on the Scantron sheet. No calculators allowed.

1. Which of the following differential equations is linear? A. $y'' + xy' + y = 1$; B. $y' + 2y^2 = 2x$; C. $(y')^2 + y^2 = 1$; D. $y' + \sin y = 0$; E. $y'y + 2y = 2x$.

2. Which of the following differential equations is separable? A. $y' + 2y = x^2$; B. $y' - xy = \cos x$; C. $xy' + y = \sin x$; D. $e^x y' + 2y = 4x$; E. $y' - x^2 y = x^2$.

3. A 10-lb weight hangs from a spring whose spring constant is 2 lb/ft. If the weight is also subject to a damping force equal to 1.5 times its velocity in ft/sec, the system is A. overdamped; B. underdamped; C. critically damped.

4. A simple circuit consists of an inductance of 1 henry, a capacitance of 0.01 farads, and a voltage source that supplies $E(t)$ volts. For which of the following functions $E(t)$ is the system in resonance? A. $20\cos t$; B. $\cos 20t$; C. $10\sin 10t$; D. $100\cos 2t$; E. $200\sin 5t$.

5. The general solution of the differential equation $y'' + 2y' + y = 0$ is A. $c_1e^{-x} + c_2e^{-x}$; B. $c_1e^{-x} + c_2e^{x}$; C. $c_1e^{-x} + c_2xe^{-x}$; D. $c_1\cos x + c_2\sin x$; E. c_1e^{-x}.

6. The general solution of the differential equation $y'' + 2y' + 5y = 0$ is A. $c_1 + c_2x$; B. $c_1\cos 2x + c_2\sin 2x$; C. $c_1e^x\cos 2x + c_2e^x\sin 2x$; D. $c_1e^{-x}\cos 2x + c_2e^{-x}\sin 2x$; E. None of these.

7. A trial solution for the inhomogeneous equation $y''+y'-2y = e^x$ is A. Ae^x; B. $Ae^x + Be^{-2x}$; C. $Ae^x + Be^{-x}$; D. Axe^x; E. None of these.

8. The inverse Laplace transform of $\frac{s}{s^2+4s+5}$ is A. $e^{-2t}\cos t$; B. $e^{-2t}\sin t$; C. $e^{2t}\cos t$; D. $e^{-2t}\cos t - 2e^{-2t}\sin t$; E. $e^t\cos 2t$.

9. The inverse Laplace transform of $\frac{2+e^{-3s}}{s^2+2s+5}$ is

   A. $e^{-t}\sin 2t + \frac{1}{2}\mathcal{U}(t-3)e^{-t}\sin 2t$;

   B. $e^{-t}\sin 2t + \frac{1}{2}\mathcal{U}(t-3)e^{3-t}\sin 2(t-3)$;

   C. $e^{-t}\cos 2t + \frac{1}{2}\mathcal{U}(t-3)e^{-t}\cos 2t$;

   D. $e^{-t}\cos 2t - \frac{1}{2}\mathcal{U}(t-3)e^{3-t}\cos 2(t-3)$;

   E. None of these.

10. The value of the function $e^t\mathcal{U}(t-2)$ at $t = 3$ is A. 0; B. e^3; C. e; D. 1; E. None of these.

11. The Laplace transform of the function $e^t\mathcal{U}(t-2)$ is A. 0; B. $\frac{1}{s-1}$; C. $\frac{e^{-2s}}{s+1}$; D. $\frac{1}{s+1}$; E. $\frac{e^{2-2s}}{s-1}$.

12. If y satisfies the initial-value problem $y'' + 9y = 4\delta(t-2)$, $y(0) = 0$, $y'(0) = 0$, then the Laplace transform $Y(s)$ of $y(t)$ is A. $\frac{4}{s^2+9}$; B. $\frac{4e^{-2s}}{s^2+9}$; C. $\frac{4e^{-2s}}{s(s^2+9)}$; D. $\frac{4e^{2s}}{s(s^2+9)}$; E. None of these.

13. The matrix $\begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix}$ has 1 as an eigenvalue. Which of the following is an eigenvector corresponding to 1? A. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$; B. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$; C. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$; D. $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$; E. None of these.

14. In the Fourier series of x^2 on the interval $(-1, 1)$, the coefficient of $\sin 3\pi x$ is A. 0; B. $\frac{2}{3\pi}$; C. $\frac{4}{9\pi^2}$; D. $-\frac{4}{9\pi^2}$; E. None of these.

15. For $x = 2$, the Fourier series of x^2 on the interval $(-1, 1)$ converges to A. 0; B. 1; C. $\frac{1}{2}$; D. $-\frac{1}{2}$; E. None of these.

16. If $u(x, y) = X(x)Y(y)$ is a product solution of the PDE $u_{xx} + u_{yy} = u_x + u_y$, then there is a constant λ so that A. $X'' = \lambda X$; B. $X'' - X' = \lambda X$; C. $X'' - X = \lambda X'$; D. $X'' = \lambda(X + X')$; E. $X'' = \lambda(X - X')$.

Part 2: Long Answer (50%)

Write your answer and supporting work in your blue book or on sheets provided. Calculators are allowed on this part.

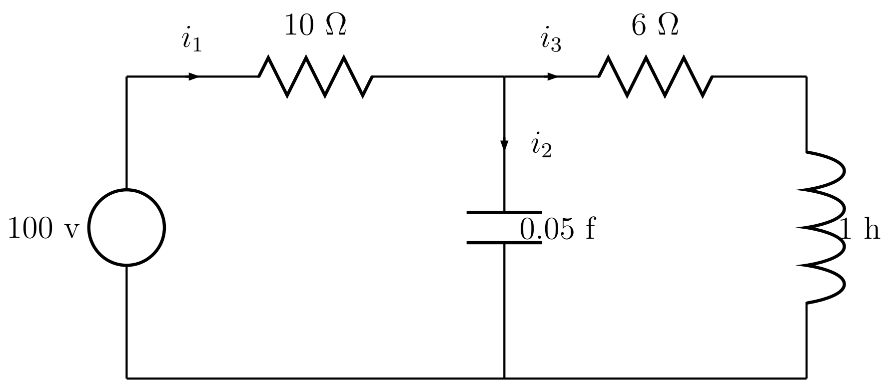
17. A cup of boiling-hot water (100° C) is placed in a room whose temperature is 20° C. After 2 minutes the water has cooled to 93° C. Assuming Newton's law of cooling, when will the temperature of the water be 30° C?

18. A 1-kg mass is attached to a spring with spring constant 1.44 Nt/m. There is no damping. Initially the mass is 1 m above the equilibrium position moving upward at 1.5 m/sec.
    (a) Let $x(t)$ be the displacement of the mass below the equilibrium position at time t seconds. Write an initial-value problem for $x(t)$.
    (b) Solve your initial-value problem to find $x(t)$ for $t \geq 0$.
    (c) What is the amplitude of the motion?

19. Solve the initial-value problem

$$y'' + 2y' + y = e^{-t}, \quad y(0) = 1, \quad y'(0) = 0.$$

20. Consider the circuit below.



    (a) Write a system of two differential equations for the current i_3 and the charge q_2 on the capacitor. (Note that $i_2 = q_2'$.)
    (b) Assuming $i_3(0) = q_2(0) = 0$, solve this system to find $i_3(t)$ and $q_2(t)$.

21. (a) Find eigenvalues and eigenvectors for the matrix

$$\begin{bmatrix} 0 & -1 \\ -9 & 0 \end{bmatrix}.$$

(b) Use your answer to Part (a) to solve the system

$$\begin{aligned} x' &= -y, \quad x(0) = 2, \\ y' &= -9x, \quad y(0) = 9. \end{aligned}$$

22. Use two steps of Euler's method to estimate $y(0.2)$, where y is the solution of the second-order initial-value problem

$$y'' - 2xy' + y = x, \quad y(0) = 1, \quad y'(0) = 1.$$

23. Let f be the function of period 6 such that

$$f(x) = \begin{cases} 0, & -3 \le x < 0 \\ x, & 0 \le x < 3 \end{cases}$$

(a) Sketch the graph of f on the interval $[-6, 6]$. At which points in this interval is f discontinuous?

(b) Find the Fourier series of f.

(c) To what value does the Fourier series converge at $x = 0$? At $x = 3$?

24. A uniform rod 2 m long has thermal diffusivity 0.4 m^2/sec. (That is, if $u(x,t)$ is the temperature at the position x meters from the left end of the rod at times t seconds, then $u_t = 0.4u_{xx}$.) Suppose the ends of the bar are perfectly insulated (so that $u_x(0,t) = u_x(2,t) = 0$ for all t), and that the initial temperature profile is given by

$$u(x, 0) = 50x, \quad 0 \le x \le 2.$$

(a) Find $u(x,t)$ for all $t > 0$.

(b) Write out explicitly the first three nonzero terms of the series solution.

SM 212 Final Exam Solutions Exam Date December 17, 2013

1. A
2. E
3. B
4. C
5. C
6. D
7. D
8. D
9. B
10. B
11. E
12. B
13. D
14. A
15. A
16. B
17. Temperature $T(t)$ satisfies IVP $T(0) = 100$, $\frac{dT}{dt} = k(T - 20)$ with solution $T(t) = 20 + 80e^{kt}$, and condition $T(2) = 93$ gives $k = \frac{1}{2} \ln \frac{73}{80}$. Solve the equation

$$20 + 80e^{\frac{t}{2} \ln \frac{73}{80}} = 30 \quad \text{to get} \quad t = 2\frac{\ln \frac{1}{8}}{\ln \frac{73}{80}} \approx 45.4 \text{ minutes}.$$

18. (a)

$$x'' + 1.44x = 0,\ x(0) = -1,\ x'(0) = -1.5.$$

(b)

$$x(t) = -\cos(1.2t) - 1.25 \sin(1.2t).$$

(c)

$$\text{amplitude} = \sqrt{1 + 1.25^2} \approx 1.601 \text{ m}.$$

19. Solution by transforms: transform the IVP to get

$$s^2Y - s + 2sY - 2 + Y = \frac{1}{s+1}, \quad \text{or} \quad (s^2 + 2s + 1)Y = \frac{s^2 + 3s + 3}{s+1},$$

hence

$$Y = \frac{s^2 + 3s + 3}{(s+1)^3} = \frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{(s+1)^3}.$$

Take inverse transforms to get

$$y = e^{-t} + te^{-t} + \frac{1}{2}t^2e^{-t}.$$

This can also be solved by undetermined coefficients.

20. (a) Left loop gives

$$10i_1 + 20q_2 = 100;$$

right loop gives

$$6i_3 + i_3' - 20q_2 = 0.$$

Use $i_1 = i_2 + i_3$ and $i_2 = q_2'$ to obtain

$$i_3' + 6i_3 - 20q_2 = 0, \quad i_3 + q_2' + 2q_2 = 10.$$

(b) Transform the system in the previous part:

$$(s+6)I_3 - 20Q_2 = 0, \quad I_3 + (s+2)Q_2 = \frac{10}{s}.$$

Solve by Cramer's rule to get

$$I_3 = \frac{\begin{vmatrix} 0 & -20 \\ \frac{10}{s} & s+2 \end{vmatrix}}{\begin{vmatrix} s+6 & -20 \\ 1 & s+2 \end{vmatrix}} = \frac{200}{s(s^2+8s+32)} = \frac{\frac{25}{4}}{s} - \frac{\frac{25}{4}(s+4)}{(s+4)^2+16} - \frac{25}{(s+4)^2+16}$$

$$Q_2 = \frac{\begin{vmatrix} s+6 & 0 \\ 1 & \frac{10}{s} \end{vmatrix}}{\begin{vmatrix} s+6 & -20 \\ 1 & s+2 \end{vmatrix}} = \frac{10(s+6)}{s(s^2+8s+32)} = \frac{\frac{15}{8}}{s} - \frac{\frac{15}{8}(s+4)}{(s+4)^2+16} + \frac{\frac{20}{8}}{(s+4)^2+16}.$$

Thus

$$\begin{aligned} i_3 &= \frac{25}{4} - \frac{25}{4}e^{-4t}\cos 4t - \frac{25}{4}e^{-4t}\sin 4t \\ q_2 &= \frac{15}{8} - \frac{15}{8}e^{-4t}\cos 4t + \frac{5}{8}e^{-4t}\sin 4t. \end{aligned}$$

21. (a) Eigenvalues are 3, -3 since

$$\begin{vmatrix} -\lambda & -1 \\ -9 & -\lambda \end{vmatrix} = (\lambda-3)(\lambda+3).$$

Eigenvectors for $\lambda = 3, -3$ are

$$\begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

respectively.

(b) The general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{3t} \begin{bmatrix} -1 \\ 3 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Initial condition says

$$c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \end{bmatrix},$$

from which follows $c_1 = \frac{1}{2}$, $c_2 = \frac{5}{2}$.

22. Let $u = y'$, so we have the system

$$y' = u, \quad u' = 2xu - y + x, \quad y(0) = 1, \quad u(0) = 1.$$

Then using step size 0.1, $y(0.1) \approx 1 + .1(1) = 1.1$ and $u(0.1) = 1 + .1(0 - 1 + 0) = 0.9$, so $y(0.2) \approx 1.1 + .1(0.9) = 1.19$.

23. (a) f is discontinuous at $x = \pm 3$.

(b) The Fourier coefficients are

$$c_0 = \frac{1}{6}\int_{-3}^{3} f(x)dx = \frac{1}{6}\int_0^3 xdx = \frac{3}{4},$$

$$a_n = \frac{1}{3}\int_{-3}^{3} f(x)\cos\frac{n\pi x}{3}dx = \frac{1}{3}\int_0^3 x\cos\frac{n\pi x}{3}dx = \frac{3}{n^2\pi^2}((-1)^n - 1),$$

$$b_n = \frac{1}{3}\int_{-3}^{3} f(x)\sin\frac{n\pi x}{3}dx = \frac{1}{3}\int_0^3 x\sin\frac{n\pi x}{3} = -\frac{3(-1)^n}{n\pi}.$$

Hence

$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty}\left[\frac{3}{n^2\pi^2}((-1)^n - 1)\cos\frac{n\pi x}{3} - \frac{3(-1)^n}{n\pi}\sin\frac{n\pi x}{3}\right].$$

(c) At $x = 0$, the series converges to 0. At $x = 3$, it converges to $\frac{3}{2}$.

24. (a) We look for a solution of the form

$$u(x,t) = c_0 + \sum_{n=1}^{\infty} a_n e^{-\frac{.4n^2\pi^2 t}{4}}\cos\frac{n\pi x}{2},$$

where to satisfy the initial condition we must have

$$c_0 = \frac{1}{2}\int_0^2 50xdx = 50$$

and

$$a_n = \int_0^2 50x\cos\frac{n\pi x}{2}dx = \frac{200}{n^2\pi^2}((-1)^n - 1).$$

Hence

$$u(x,t) = 50 + \sum_{n=1}^{\infty}\frac{200}{n^2\pi^2}((-1)^n - 1)e^{-.1n^2\pi^2 t}\cos\frac{n\pi x}{2}.$$

(b) The series begins

$$u(x,t) = 50 - \frac{400}{\pi^2}e^{-.1\pi^2 t}\cos\frac{\pi x}{2} - \frac{400}{9\pi^2}e^{-.9\pi^2 t}\cos\frac{3\pi x}{2} - \cdots$$