SM 212 Final Exam — Part I (no calculator)

Saturday, 3 May 2014, 1330-1630

Identification

Name: _____

Alpha: _____

Section: _____

Instructor: _____

Instructions (read carefully)

1. The exam consists of two parts. This is Part I.
2. Write your name and alpha on both parts of the exam, as well as on the scantron form provided. Also, bubble in your alpha on the scantron form using a #2 pencil.
3. Part I has 20 multiple-choice problems. Record your answers on the scantron form, using a #2 pencil. There is no penalty for guessing.
4. Part II has 8 free-response problems. Record your answers, including all your work, on the exam itself.
5. You may use the approved note sheet throughout the exam.
6. **You may only start using the calculator once you turn in Part I and your scantron form.** Use of the calculator is limited to computations. It should not be used to store additional notes.
7. No other resources are allowed (books, notes, neighbor, cell-phone, etc.).
8. If you need additional scratch paper, ask your instructor.
9. If you need to use the head, ask your instructor for permission. Leave your cell-phone in the exam room, and place your exam materials face down on your desk.

Part I: Multiple Choice (no calculator)

Question 1. *Classify the following differential equation:*

$$x\frac{d^3y}{dx^3} - \left(\frac{dy}{dx}\right)^4 + y = 0$$

(a) 3rd-order, linear.

(b) 3rd-order, non-linear.

(c) 4th-order, linear.

(d) 4th-order, non-linear.

Question 2. *Given that $y_p = \ln(x)$ is a particular solution of the non-homogeneous equation*

$$x^2y'' + xy' + y = \ln(x)$$

and that $y_1 = \cos(\ln(x))$ and $y_2 = \sin(\ln(x))$ are linearly independent solutions of the associated homogeneous equation

$$x^2y'' + xy' + y = 0,$$

which of the following is the <u>general solution</u> of the non-homogenous equation?

(a) $y = c_1\cos(\ln(x)) + c_2\sin(\ln(x)) + \ln(x)$

(b) $y = c_1\cos(\ln(x)) + c_2\sin(\ln(x)) + c_3\ln(x)$

(c) $y = c_1\cos(\ln(x)) + c_2\sin(\ln(x))$

(d) $y = c_3\ln(x)$

Question 3. *For which values of m is the function $y = x^m$ a solution of the differential equation*

$$x^2y'' - 5xy' + 8y = 0?$$

(a) $m = 2, 3$

(b) $m = 2, 4$

(c) $m = 3, 4$

(d) $m = -2, -4$

Question 4. *For the 1st-order linear differential equation*

$$xy' + 4y = x^2,$$

what is the integrating factor?

(a) $\mu(x) = e^{4x}$

(b) $\mu(x) = x^4$

(c) $\mu(x) = 4\ln(x)$

(d) $\mu(x) = e^{4/x}$

Question 5. *A large tank originally contains* 200 *gal of water with* 100 *lb of salt in solution. Water containing* 1 *lb/gal of salt is entering at a rate of* 3 *gal/min, and the mixture is allowed to flow out of the tank at a rate of* 2 *gal/min. If* $A(t)$ *is the amount of salt (lb) in the tank at time* t *(min), then* $A(t)$ *satisfies which of the following initial value problems?*

(a) $\frac{dA}{dt} = 3 - \frac{2A}{200+t}$, $A(0) = 1/2$

(b) $\frac{dA}{dt} = 3 - \frac{A}{100}$, $A(0) = 1/2$

(c) $\frac{dA}{dt} = 3 - \frac{2A}{200+3t}$, $A(0) = 100$

(d) $\frac{dA}{dt} = 3 - \frac{2A}{200+t}$, $A(0) = 100$

Question 6. *A* 1 *kg mass is attached to a spring with constant* 13 *N/m. The system is immersed in a medium which offers a damping force numerically equal to* 6 *times the instantaneous velocity. If* x *is the displacement of the mass from equilibrium, measured in meters, then*

$$x'' + 6x' + 13x = 0.$$

Which of the following statements is true?

(a) $x(t) = c_1e^{-t} + c_2e^{-5t}$, *and the system is underdamped.*

(b) $x(t) = c_1e^{-t} + c_2e^{-5t}$, *and the system is overdamped.*

(c) $x(t) = c_1e^{-3t}\cos(2t) + c_2e^{-3t}\sin(2t)$, *and the system is underdamped.*

(d) $x(t) = c_1e^{-3t}\cos(2t) + c_2e^{-3t}\sin(2t)$, *and the system is overdamped.*

Question 7. *Given that*

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 2 & -1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

what is the solution of the following system of linear equations?

$$\begin{cases} x + 2y - z = 2 \\ 2x + y + z = 1 \\ x - y + 2z = -1 \end{cases}$$

(a) $x = 0,\ y = 1,\ z = 0.$

(b) *The system has no solution.*

(c) $x = -t,\ y = 1 + t,\ z = t$ *(t arbitrary).*

(d) $x = -t,\ y = 1 + t,\ z = 0$ *(t arbitrary).*

Question 8. *Let $y_1 = e^{2t}$ and $y_2 = te^{2t}$. Which of the following statements is true about $W(y_1, y_2)$, the Wronskian of y_1 and y_2?*

(a) $W(y_1, y_2) = e^{4t}$, *therefore y_1 and y_2 are linearly independent.*

(b) $W(y_1, y_2) = e^{4t}$, *therefore y_1 and y_2 are linearly dependent.*

(c) $W(y_1, y_2) = 0$, *therefore y_1 and y_2 are linearly independent.*

(d) $W(y_1, y_2) = 0$, *therefore y_1 and y_2 are linearly dependent.*

Question 9. *Consider the 2nd-order non-homogeneous differential equation*

$$y'' - 4y' + 3y = e^t + t^2.$$

What is the complementary (or homogeneous) solution?

(a) $y_c = c_1 e^t + c_2 t^2$

(b) $y_c = c_1 e^{-t} + c_2 e^{-3t}$

(c) $y_c = c_1 e^t + c_2 e^{3t}$

(d) $y_c = c_1 e^t + c_2 e^{-3t}$

Question 10. *For the differential equation in the previous question, what is the correct form for a particular solution?*

(a) $y_p = Ae^t + Bt^2$

(b) $y_p = Ae^t + Bt^2 + Ct + D$

(c) $y_p = Ate^t + Bt^2$

(d) $y_p = Ate^t + Bt^2 + Ct + D$

Question 11. *The Laplace transform of the function* $f(t) = (t+2)^2$ *is:*

(a) $\left(\frac{1}{s^2} + \frac{2}{s}\right)^2$

(b) $\frac{2}{s^3} + \frac{4}{s}$

(c) $\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}$

(d) $\frac{2}{(s+2)^3}$

Question 12. *The Laplace transform of the function* $f(t) = t\sin(t)$ *is:*

(a) $\frac{2s}{(s^2+1)^2}$

(b) $\frac{1}{s^2(s^2+1)}$

(c) $\frac{1}{s^2} + \frac{1}{s^2+1}$

(d) $-\frac{2s}{(s^2+1)^2}$

Question 13. *The inverse Laplace transform of the function*

$$F(s) = \frac{e^{-2s}s}{s^2+4}$$

is:

(a) $\frac{1}{2}\sin(2t-4)\mathcal{U}(t-2)$

(b) $\cos(2t-2)\mathcal{U}(t-2)$

(c) $\cos(2t-4)\mathcal{U}(t-2)$

(d) $\delta(t-2)\cos(2t)$

Question 14. *Which of the following statements is true about* $t * t$*, the convolution of* t *with* t*?*

(a) $t * t = \int_0^t \tau(t-\tau)d\tau$.

(b) $\mathcal{L}\{t * t\} = \frac{1}{s^4}$.

(c) $t * t = \frac{t^3}{6}$.

(d) *All of the above.*

Question 15. *In terms of unit step functions, the piecewise function*

$$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ -1, & 1 \le t < 2 \\ 0, & t \ge 2 \end{cases}$$

can be expressed as:

(a) $f(t) = 1 - \mathcal{U}(t-1)$

(b) $f(t) = 1 - \mathcal{U}(t-1) + \mathcal{U}(t-2)$

(c) $f(t) = 1 - 2\mathcal{U}(t-1) + 2\mathcal{U}(t-2)$

(d) $f(t) = 1 - 2\mathcal{U}(t-1) + \mathcal{U}(t-2)$

Question 16. *Which of the following statements about*

$$A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$$

is true?

(a) $A^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 1/4 \end{bmatrix}$

(b) $A^{-1} = \begin{bmatrix} 2 & -1 \\ -3/2 & 1 \end{bmatrix}$

(c) A^{-1} *does not exist.*

(d) $A^{-1} = \begin{bmatrix} 4 & -2 \\ -3 & 2 \end{bmatrix}$

Question 17. *The eigenvalues of the matrix* $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ *are:*

(a) $\lambda = \pm 1$

(b) $\lambda = 0, 2$

(c) $\lambda = \pm i$

(d) $\lambda = 1 \pm i$

Question 18. *Given that the eigenvalues of the matrix* $A = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}$ *are* $\lambda = 2, -1$, *with corresponding eigenvectors* $\vec{K} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, *what is the general solution of the following system of differential equations?*

$$\begin{cases} \frac{dx}{dt} = 3x - y \\ \frac{dy}{dt} = 4x - 2y \end{cases}$$

(a) $x = c_1 e^{2t} + c_2 e^{-t}$, $y = c_1 e^{2t} + 4c_2 e^{-t}$

(b) $x = c_1 + c_2$, $y = c_1 + 4c_2$

(c) $x = c_1 e^{2t}$, $y = c_2 e^{-t}$

(d) $x = e^{2t} + e^{-t}$, $y = e^{2t} + 4e^{-t}$

Question 19. *Let* $f(x) = |x|$, $-\pi < x < \pi$, *and let*

$$FS(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

be its Fourier series. Which of the following statements is true?

(a) $a_0 = 0$.

(b) $a_n = 0$ *for* $n = 1, 2, 3, \ldots$.

(c) $b_n = 0$ *for* $n = 1, 2, 3, \ldots$.

(d) Both (a) and (b).

Question 20. *The partial differential equation*

$$u_{xx} + u_{yy} = 0$$

is called Laplace's equation. Which of the following functions is a <u>*product solution*</u> *of Laplace's equation?*

(a) $u(x, y) = x^2 - y^2$

(b) $u(x, y) = e^x \sin(y)$

(c) $u(x, y) = x^2 y^2$

(d) All of the above.

SM 212 Final Exam — Part II

Saturday, 3 May 2014, 1330-1630

Identification

Name: _____

Alpha: _____

Section: _____

Instructor: _____

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Part II: Free Response

(calculator allowed, once you turn in Part I and scantron)

Question 1. *Solve the following 1st-order differential equations:*

(a) $\frac{dy}{dx} = y^2 \sin(x)$.

(b) $\frac{dy}{dx} + \frac{y}{x} = \cos(x)$.

Question 2. *Use* <u>*undetermined coefficients*</u> *to solve the following initial value problem:*

$$\begin{cases} y'' + 2y' + y = 2e^{-t} \\ y(0) = 1 \\ y'(0) = 0 \end{cases} .$$

Question 3. *Use* <u>*Laplace transforms*</u> *to solve the following initial value problem (same one as in the previous question):*

$$\begin{cases} y'' + 2y' + y = 2e^{-t} \\ y(0) = 1 \\ y'(0) = 0 \end{cases} .$$

Question 4. *Use* <u>*Euler's method*</u> *with a step size of* $h = 0.1$ *to estimate* $y(0.3)$ *to* <u>*four decimal places*</u>*, where* $y(t)$ *is the solution of the following initial value problem (same one as in the previous two questions):*

$$\begin{cases} y'' + 2y' + y = 2e^{-t} \\ y(0) = 1 \\ y'(0) = 0 \end{cases} .$$

Remark: *Since you found a formula for* $y(t)$ *in the previous two questions, you can compute* $y(0.3)$ *exactly to check your Euler's method estimate.*

Question 5. *Solve the following system using* <u>*Laplace transforms*</u>*:*

$$\begin{cases} \frac{dx}{dt} = 2x - y + t \\ \frac{dy}{dt} = 3x - 2y - 1 \\ x(0) = 0,\ y(0) = 0 \end{cases} .$$

Question 6. *Solve the following system (same one as in the previous question) by rewriting it as a matrix differential equation, and using eigenvalues and eigenvectors:*

$$\begin{cases} \frac{dx}{dt} = 2x - y + t \\ \frac{dy}{dt} = 3x - 2y - 1 \\ x(0) = 0,\ y(0) = 0 \end{cases} .$$

Question 7. *Let*

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 \leq x < \pi \end{cases}.$$

(a) *Find $FS(x)$, the Fourier series for $f(x)$.*

(b) *Graph $FS(x)$ for $-3\pi \leq x \leq 3\pi$.* ***Remark:*** *Don't try to do this with the calculator. Rather, think about how the graph of $FS(x)$ relates to the graph of $f(x)$.*

Question 8. *Find all* <u>*product solutions*</u> *of the following partial differential equation:*

$$\frac{1}{x}u_x + \frac{1}{y}u_y = u.$$

SM 212 Final Exam — Answer Key

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Multiple Choice

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1. (b)

Solution. The equation is third-order because the highest derivative of the unknown function y is $\frac{d^3y}{dx^3}$. The equation is non-linear because of the term $\left(\frac{dy}{dx}\right)^4$. □

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2. (a)

Solution. The general solution of a linear 2nd-order equation is $y = c_1y_1 + c_2y_2 + y_p$, where y_p is a particular solution and y_1 and y_2 are linearly independent solutions of the associated homogeneous equation. □

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3. (b)

Solution.

$$y = x^m \Rightarrow y' = mx^{m-1} \to y'' = m(m-1)x^{m-2}$$

$$\begin{aligned} x^2y'' - 5xy' + 8y = 0 \quad &\Rightarrow \quad m(m-1)x^m - 5mx^m + 8x^m = 0 \\ &\Rightarrow \quad (m^2 - 6m + 8)x^m = 0 \\ &\Rightarrow \quad (m-2)(m-4) = 0 \\ &\Rightarrow \quad m = 2, 4 \end{aligned}$$

□

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4. (b)

Solution. Standard form:

$$y' + \frac{4}{x}y = x$$

Integrating factor:

$$\mu(x) = e^{\int (4/x)dx} = e^{4\ln(x)} = e^{\ln(x^4)} = x^4$$

□

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5. (d)

Solution.

$$\frac{dA}{dt} = (3)(1) - (2)\left(\frac{A}{200 + (3-2)t}\right), \ A(0) = 100$$

$$\frac{dA}{dt} = 3 - \frac{2A}{200 + t}, \ A(0) = 100$$

□

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6. (c)

Solution. Auxiliary equation:

$$r^2 + 6r + 13 = 0$$

$$r = \frac{-6 \pm \sqrt{36 - 52}}{2} = \frac{-6 \pm 4i}{2} = -3 \pm 2i$$

Solution:

$$x(t) = c_1 e^{-3t}\cos(2t) + c_2 e^{-3t}\sin(2t)$$

Underdamped because the solution is oscillatory with decaying amplitudes.

□

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7. (c)

Solution. From the reduced row echelon form, the original system has the same solution as the following system:

$$\begin{cases} x + z = 0 \\ y - z = 1 \end{cases}$$

Setting $z = t$ and solving for x and y yields $x = -t$ and $y = 1 + t$. □

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8. (a)

Solution.

$$W(y_1, y_2) = \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & (2t+1)e^{2t} \end{vmatrix} = (2t+1)e^{4t} - 2te^{4t} = e^{4t} \neq 0$$

Therefore the functions are linearly independent. □

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9. (c)

Solution.

$$r^2 - 4r + 3 = 0 \Rightarrow (r-1)(r-3) = 0 \Rightarrow r = 1, 3$$

$$y_c = c_1 e^t + c_2 e^{3t}$$

□

10. (d)

Proof. By the multiplication rule and the superposition rule:

$$y_p = Ate^t + Bt^2 + Ct + D$$

In fact,

$$y_p = -\frac{1}{2}te^t + \frac{1}{3}t^2 + \frac{8}{9}t + \frac{26}{27}$$

□

11. (c)

Solution.

$$\begin{aligned}
\mathcal{L}\{(t+2)^2\} &= \mathcal{L}\{t^2 + 4t + 4\} \\
&= \frac{2!}{s^3} + 4\left(\frac{1}{s^2}\right) + 4\left(\frac{1}{s}\right) \\
&= \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}
\end{aligned}$$

□

12. (a)

Solution.

$$\begin{aligned}
\mathcal{L}\{t\sin(t)\} &= -\frac{d}{ds}\mathcal{L}\{\sin(t)\} = -\frac{d}{ds}\left[\frac{1}{s^2+1}\right] \\
&= -\left[-\frac{2s}{(s^2+1)^2}\right] = \frac{2s}{(s^2+1)^2}
\end{aligned}$$

□

13. (c)

Solution.

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{e^{-2s}s}{s^2+4}\right\} &= \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}|_{t\to t-2}\mathcal{U}(t-2) \\
&= \cos(2t)|_{t\to t-2}\mathcal{U}(t-2) \\
&= \cos(2(t-2))\mathcal{U}(t-2) \\
&= \cos(2t-4)\mathcal{U}(t-2)
\end{aligned}$$

□

14. (d)

Solution.

$$\mathcal{L}\{t * t\} = \mathcal{L}\{t\}\mathcal{L}\{t\} = \left(\frac{1}{s^2}\right)\left(\frac{1}{s^2}\right) = \frac{1}{s^4}$$

$$t * t = \mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{1}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} = \frac{t^3}{6}$$

$$\begin{aligned} t * t &= \int_0^t \tau(t-\tau)d\tau = \int_0^t (t\tau - \tau^2)d\tau \\ &= \left[\frac{t\tau^2}{2} - \frac{\tau^3}{3}\right]_0^t = \frac{t^3}{2} - \frac{t^3}{3} = \frac{t^3}{6} \end{aligned}$$

□

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15. (d)

Solution.

$$f(t) = 1 - 2\mathcal{U}(t-1) + \mathcal{U}(t-2)$$

□

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16. (b)

Solution.

$$A^{-1} = \frac{1}{(2)(4)-(2)(3)}\begin{bmatrix} 4 & -2 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3/2 & 1 \end{bmatrix}$$

□

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17. (b)

Solution.

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1 \\ &= \lambda^2 - 2\lambda = \lambda(\lambda - 2). \end{aligned}$$

$$\det(A - \lambda I) = 0 \Rightarrow \lambda = 0, 2$$

□

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18. (a)

Solution. Note that the given system of differential equations is equivalent to the matrix differential equation

$$\frac{d\vec{X}}{dt} = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}\vec{X}.$$

Using the given information about the eigenvalues and eigenvectors of A, the general solution of the matrix differential equation is

$$\vec{X} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-t}.$$

Breaking the general solution into its component functions,

$$x = c_1 e^{2t} + c_2 e^{-4t}, \ y = c_1 e^{2t} + 4c_2 e^{-t}.$$

□

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19. (c)

Solution. Since f is even, $b_n = 0$ for all $n = 1, 2, 3, ...$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x dx = \pi \neq 0$$

If $a_n = 0$ for all $n = 1, 2, 3, ...,$ then f would be a constant, which it isn't. □

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20. (b)

Solution. If $u(x, y) = x^2 - y^2$, then

$$u_{xx} + u_{yy} = 2 + (-2) = 0,$$

so $u(x, y)$ is a solution of Laplace's equation, but not a product solution. If $u(x, y) = e^x \sin(y)$, then

$$u_{xx} + u_{yy} = e^x \sin(y) + (-e^x \sin(y)) = 0,$$

so $u(x, y)$ is a product solution of Laplace's equation. If $u(x, y) = x^2 y^2$, then

$$u_{xx} + u_{yy} = 2y^2 + 2x^2,$$

so $u(x, y)$ is not a solution of Laplace's equation. □

Free Response

1.

Solution. (a) Separation of variables:

$$\int y^{-2} dy = \int \sin(x) dx$$

$$-y^{-1} = -\cos(x) + C$$

$$y^{-1} = \cos(x) - C$$

$$y = \frac{1}{\cos(x) - C}$$

(b) Integrating factor:

$$\mu(x) = e^{\int (1/x) dx} = e^{\ln(x)} = x$$

$$xy' + y = x\cos(x)$$

$$(xy)' = x\cos(x)$$

$$xy = x\sin(x) + \cos(x) + C$$

$$y = \sin(x) + \frac{\cos(x)}{x} + \frac{C}{x}$$

$\square$

2.

Solution. Homogeneous equation:

$$y'' + 2y' + y = 0$$

Auxiliary equation:

$$r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r = -1, -1$$

Complementary solution:

$$y_c = c_1 e^{-t} + c_2 t e^{-t}$$

Form of particular solution:

$$y_p = At^2 e^{-t}$$

Undetermined coefficients:

$$y_p' = A(2t - t^2)e^{-t}$$

$$y_p'' = A(2 - 4t + t^2)e^{-t}$$

$$A(2 - 4t + t^2)e^{-t} + 2A(2t - t^2)e^{-t} + At^2 e^{-t} = 2e^{-t}$$

$$2Ae^{-t} = 2e^{-t} \Rightarrow A = 1$$

$$y_p = t^2 e^{-t}$$

General solution:

$$y = c_1 e^{-t} + c_2 t e^{-t} + t^2 e^{-t}$$

Initial conditions:

$$1 = y(0) = c_1$$

$$y' = -c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t} + 2te^{-t} - t^2 e^{-t}$$

$$0 = y'(0) = -c_1 + c_2 \Rightarrow c_2 = 1$$

Solution:

$$y = e^{-t} + te^{-t} + t^2 e^{-t} = (t^2 + t + 1)e^{-t}$$

$\square$

3.

Solution.

$$(s^2Y - s - 0) + 2(sY - 1) + Y = \frac{2}{s+1}$$

$$(s^2 + 2s + 1)Y = s + 2 + \frac{2}{s+1}$$

$$Y = \frac{s^2 + 3s + 4}{(s+1)^3} = \frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{2}{(s+1)^3}$$

$$\begin{aligned} y &= \mathcal{L}^{-1}\left\{\frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{2}{(s+1)^3}\right\} \\ &= e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{1}{s^2} + \frac{2}{s^3}\right\} \\ &= e^{-t}(1 + t + t^2) \end{aligned}$$

$\square$

4.

Solution. Convert 2nd-order equation to system of two 1st-order equations:

$$\begin{cases} u = y' \\ u' + 2u + y = 2e^{-t} \\ y(0) = 1,\ u(0) = 0 \end{cases}$$

Re-organize:

$$\begin{cases} \frac{dy}{dt} = u \\ \frac{du}{dt} = 2e^{-t} - 2u - y \\ y(0) = 1,\ u(0) = 0 \end{cases}$$

Euler's method:

t	y	u	Δt	$\frac{dy}{dt} = u$	$\Delta y \approx \frac{dy}{dt}\Delta t$	$\frac{du}{dt} = 2e^{-t} - 2u - y$	$\Delta u \approx \frac{du}{dt}\Delta t$
0	1	0	0.1	0	0	1	0.1
0.1	1	0.1	0.1	0.1	0.01	0.6097	0.0610
0.2	1.01	0.1610	0.1	0.1610	0.0161	$\times$	$\times$
0.3	1.0261	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$

$$y(1) \approx 1.0261$$

Actual value:

$$y(0.3) = (0.3^2 + 0.3 + 1)e^{-0.3} \approx 1.0297$$

$\square$

5.

Solution. Laplace transform:

$$\begin{cases} sX = 2X - Y + \frac{1}{s^2} \\ sY = 3X - 2Y - \frac{1}{s} \end{cases}$$

Re-organize:

$$\begin{cases} (s-2)X + Y = \frac{1}{s^2} \\ -3X + (s+2)Y = -\frac{1}{s} \end{cases}$$

Solve system:

$$\begin{bmatrix} s-2 & 1 & \frac{1}{s^2} \\ -3 & s+2 & -\frac{1}{s} \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & \frac{2}{s^2(s^2-1)} \\ 0 & 1 & -\frac{s-3}{s^2(s^2-1)} \end{bmatrix}$$

$$X = \frac{2}{s^2(s^2-1)} = \frac{2}{s-1} - \frac{2}{s} - \frac{2}{s^2}$$

$$Y = -\frac{s^2-3}{s^2(s^2-1)} = \frac{2}{s-1} - \frac{2}{s} - \frac{3}{s^2}$$

Invert:

$$x = 2e^t - 2 - 2t$$

$$y = 2e^t - 2 - 3t$$

$\square$

6.

Solution. Matrix form:

$$\frac{d\vec{X}}{dt} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{X} + \begin{bmatrix} t \\ -1 \end{bmatrix}$$

Eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

Eigenvectors for $\lambda = 1$:

$$A\vec{K} = 1\vec{K} \iff (A - 1I)\vec{K} = \vec{0} \iff \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x - y = 0 \Rightarrow x = t,\ y = t \Rightarrow \vec{K} = \begin{bmatrix} t \\ t \end{bmatrix} \overset{(t=1)}{=} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Eigenvectors for $\lambda = -1$:

$$A\vec{K} = -1\vec{K} \iff (A - (-1)I)\vec{K} = \vec{0} \iff \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3x - y = 0 \Rightarrow x = t,\ y = 3t \Rightarrow \vec{K} = \begin{bmatrix} t \\ 3t \end{bmatrix} \overset{(t=1)}{=} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Complementary solution:

$$\vec{X}_c = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$$

Particular solution (variation of parameters):

$$\vec{X}_p = \Phi(t) \int \Phi^{-1}(t)\vec{F}(t)dt = \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix} \int \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix}^{-1} \begin{bmatrix} t \\ -1 \end{bmatrix} dt = \begin{bmatrix} -2t-2 \\ -3t-2 \end{bmatrix}$$

General solution:

$$\vec{X} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t} + \begin{bmatrix} -2t-2 \\ -3t-2 \end{bmatrix}$$

Initial conditions:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{X}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$c_1 = 2,\ c_2 = 0$$

Solution:

$$\vec{X} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} e^t + \begin{bmatrix} -2t-2 \\ -3t-2 \end{bmatrix} \Rightarrow x = 2e^t - 2t - 2,\ y = 2e^t - 3t - 2$$

$\square$

7.

Solution. Fourier series:

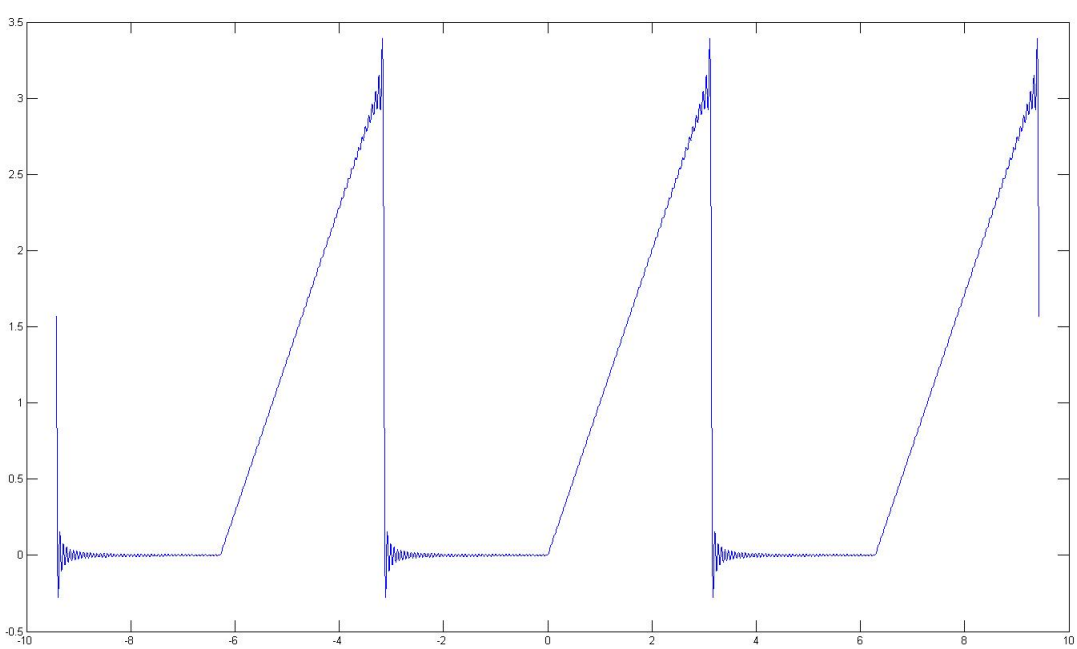
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{0}^{\pi} xdx = \frac{\pi}{2}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)dx \\ &= \frac{\cos(n\pi) + n\sin(n\pi)\pi - 1}{n^2\pi} \\ &= \frac{(-1)^n - 1}{n^2\pi} \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\int_{0}^{\pi} x\sin(nx)dx \\ &= -\frac{n\cos(n\pi)\pi - \sin(n\pi)}{n^2\pi} \\ &= -\frac{(-1)^n}{n}. \end{aligned}$$

$$FS(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2\pi}\cos(nx) - \frac{(-1)^n}{n}\sin(nx)\right]$$

Graph (approximate):



□

8.

Solution.

$$u = XY$$

$$\frac{1}{x}X'Y + \frac{1}{y}XY' = XY$$

$$\frac{X'}{xX} + \frac{Y'}{yY} = 1$$

$$\frac{X'}{xX} = 1 - \frac{Y'}{yY} = \lambda$$

$$\frac{X'}{xX} = \lambda \Rightarrow \frac{X'}{X} = \lambda x \Rightarrow \ln(X) = \frac{\lambda x^2}{2} + C \Rightarrow X = C_1 e^{\lambda x^2/2}$$

$$1 - \frac{Y'}{yY} = \lambda \Rightarrow \frac{Y'}{Y} = (1-\lambda)y \Rightarrow \ln(Y) = \frac{(1-\lambda)y^2}{2} + D \Rightarrow Y = D_1 e^{(1-\lambda)y^2/2}$$

$$u = Ae^{(\lambda x^2 + (1-\lambda)y^2)/2}$$

$\square$