NAME:____________KEY______________ ALPHA NUM:___________
INSTRUCTOR: ___________________________ SECTION: ______________

CALCULUS III (SM221,SM221P) FINAL EXAMINATION

1330-1630 14 Dec 2010 SHOW ALL WORK IN THIS TEST PACKAGE

Tear off the last sheet of this exam. It contains problems 27, 28, 29, 30. Do these problems first WITHOUT YOUR CALCULATOR. When you are finished with these 4 problems, return the sheet to your instructor, take out your calculator, and complete the rest of the exam. For all of the exam you should have as a formula sheet a copy of page 1105 from the text.

PART ONE: MULTIPLE CHOICE (50%). The first 20 problems are multiple-choice. Fill in the best answer on your Scantron bubble sheet. Write your name, alpha number, instructor, and section number on this test and your bubble sheet and bubble in your alpha number. There is no extra penalty for wrong answers on the multiple choice. Show all your scratch work on this test.

CALCULATORS ARE PERMITTED FOR THIS MULTIPLE CHOICE SECTION.

---

1. For $g(x,y,z) = 3x^2y - y^2z + z$, $g_y(1,2,3) =$

a. -9 b. -8 c. -3 d. 0 e. 4

$g_y(x,y,z) = 3x^2 - 2yz$ so $g_y(1,2,3) = 3 - 12 = -9$. So [a].

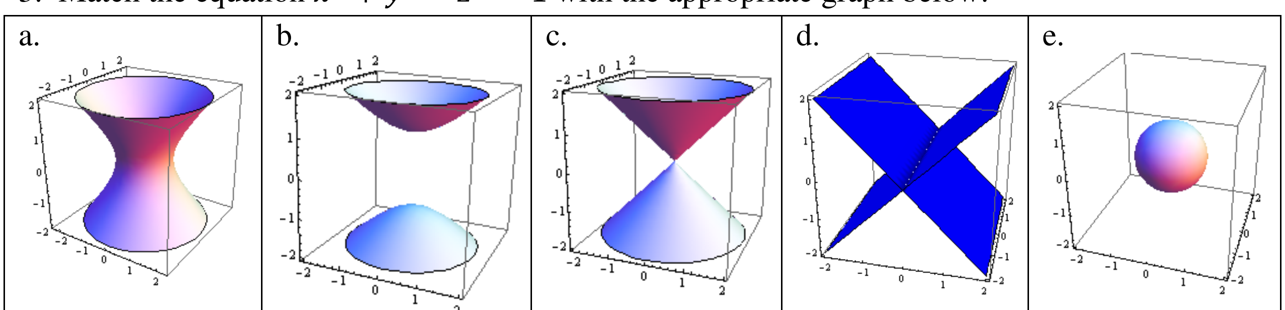
---

2. Which equation below describes a line PARALLEL to the plane with equation $3x - 2y + z = 7$?

a. $\mathbf{r}(t) = \langle 1+3t, -2t, 2+t\rangle$
b. $\mathbf{r}(t) = \langle 1+2t, 3t, 2\rangle$
c. $\mathbf{r}(t) = \langle 1+t, t, 2+t\rangle$
d. $\mathbf{r}(t) = \langle 1+t/3, -t/2, 2+t\rangle$
e. $\mathbf{r}(t) = \langle 7+t, 7+t, 7+t\rangle$

The normal direction to the plane is given by $\langle 3,-2,1\rangle$. In order for a line's direction to be parallel to the plane, it must be perpendicular to the normal. Only b. satisfies this; it's direction is $\langle 2,30\rangle$ with $\langle 3,-2,1\rangle \cdot \langle 2,3,0\rangle = 0$. So [b] .

3. Match the equation $x^2 + y^2 - z^2 = 1$ with the appropriate graph below:



[a] For example, horizontal slices with $z = K$ all give circles with equations $x^2 + y^2 = 1 + K^2$ and radius at least 1.

4. Which of the following is a tangent vector to the curve given by $\mathbf{r}(t) = \ln(t)\mathbf{i} - 2\sqrt{t}\mathbf{j} + 3\mathbf{k}$ at the point $(0, -2,3)$?

a. $\mathbf{i} - \mathbf{j}$ b. $-2\mathbf{j} + 3\mathbf{k}$ c. $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ d. $-\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$ e. does not exist

The curve goes through $(0, -2,3)$ for $t = 1$ since $\mathbf{r}(1) = \langle 0, -2,3\rangle$. So a tangent vector is given by $\mathbf{r}'(t) = \langle \frac{1}{t}, -t^{-\frac{1}{2}}, 0\rangle$ for $t = 1$, so $\langle 1, -1,0\rangle$ or $\boxed{a}$.

5. Find, to 3 digit accuracy, the length of the curve in space (a part of a parabola) given by $\mathbf{r}(t) = \langle t, t^2, 3\rangle$ that goes from $(0,0,3)$ to $(1,1,3)$.

a. 1.00 b. 1.40 c. 1.48 d. 1.51 e. 2.33

Note that the limits on t are 0 and 1 since $\mathbf{r}(0) = \langle 0,0,3\rangle$ and $\mathbf{r}(1) = \langle 1,1,3\rangle$. So $L = \int_0^1 |\mathbf{r}'(t)|\, dt = \int_0^1 \sqrt{(1)^2 + (2t)^2 + (0)^2}\ dt = \int_0^1 \sqrt{1 + 4t^2}\ dt \cong 1.47894$. Hence $\boxed{c}$.

6. Which correctly describes both the domain (the biggest set on which the formula makes sense) and the range of the function given by $f(x, y) = \sqrt{y - x + 2}$?

a. The domain is a region on and **above** a line and the range is the interval $[0, \infty)$.
b. The domain is a region on and **below** a line and the range is the interval $(-\infty, \infty)$.
c. The domain is a region on and **below** a line and the range is the interval $(-\infty, 0)$.
d. The domain is a region on and **below** a line and the range is the interval $[0, \infty)$.
e. The domain is a region on and **above** a line and the range is the interval $(-\infty, \infty)$.

Since we can't take the square root of a negative number, we must have, $y - x + 2 \geq 0$ so $y \geq x - 2$ which is satisfied for (x, y) on or above the line $y = x - 2$. The output of the square root is never negative, but we can find (x, y) to get any other output. So the range is $[0, \infty)$. $\boxed{a}$.

7. For $h(x, y) = \sin(2x) + \cos(3x) + e^{x+y}$, the linear approximation at $(0,0)$ is given by $L(x, y) =$

a. $2 - 3x + y$ b. $3x + y$ c. $\langle 3,1\rangle$
d. $2 + 3x + y$ e. $2 + (2\cos(2x) - 3\sin(3x) + e^{x+y})x + (e^{x+y})y$

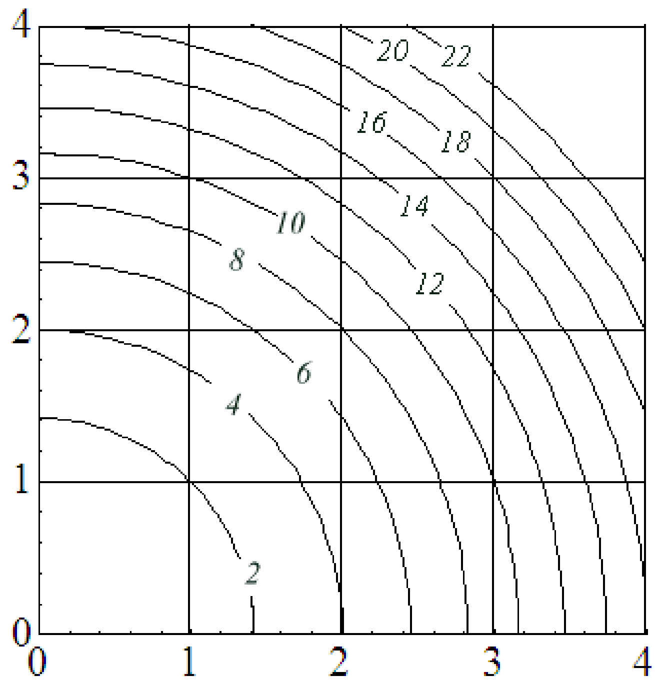
Since $h(0,0) = 2$, $h_x(x, y) = 2\cos(2x) - 3\sin(3x) + e^{x+y}$, $h_x(0,0) = 3$, $h_y(x, y) = e^{x+y}$, and $h_y(0,0) = 1$. So $(x, y) = 2 + 3(x - 0) + 1(y - 0) = 2 + 3x + y$. So $\boxed{d}$.

8. If the gradient of f at the origin is given by $\nabla f(0,0) = \langle -2,3\rangle$, then the fastest rate of increase of f at the origin is:

a. 1 b. $\sqrt{5}$ c. $\sqrt{13}$ d. 5 e. 13

The fastest rate of increase is the magnitude of the gradient or $|\langle -2,3\rangle| = \sqrt{13}$, or $\boxed{c}$.

9. A contour map is shown for a function f on the square $R = [0,4] \times [0,4]$. Use the Midpoint Rule with $m = n = 2$ to approximate $\iint_R f(x,y)dA$.



a. 26

b. 40

c. 80

d. 120

e. 160

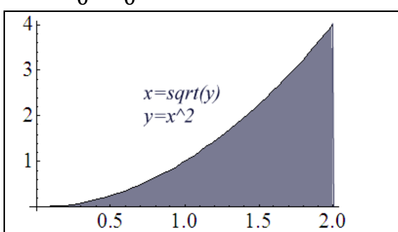
The subrectangles are 2×2 squares with centers at $(1,1), (3,1), (1,3), (3,3)$ where the function values are $2, 10, 10, 18$ respectively. So the approximation is the sum of these values times 4 (the area of each subrectangle). So we get $(2 + 10 + 10 + 18)4 = 160$. So e .

---

10. Which of the following correctly reverses the order of integration for $\int_0^4 \int_{\sqrt{y}}^2 f(x,y)dxdy$?

a. $\int_0^2 \int_0^{x^2} f(x,y)dydx$ b. $\int_0^4 \int_0^{x^2} f(x,y)dydx$ c. $\int_{\sqrt{y}}^2 \int_0^4 f(x,y)dydx$

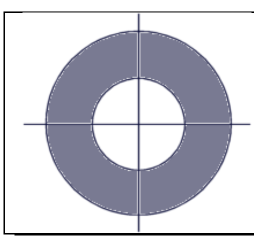
d. $\int_0^2 \int_0^{\sqrt{x}} f(x,y)dydx$ e. $\int_0^2 \int_{x^2}^4 f(x,y)dydx$



In the y-direction, we go from bottom curve $y = 0$ to top curve $y = x^2$, then from the left point $x = 0$ to the right point $x = 2$. So this gives us a $\int_0^2 \int_0^{x^2} f(x,y)dydx$.

11. Find the mass of a lamina that lies between the circles centered at the origin given by $r = 1$ and $r = 2$ if the density equals the reciprocal of distance from the origin.

a. π b. 2π c. 3π d. 4π e. $2\pi \ln(2)$



In polar coordinates density equals $1/r$ so mass is given by $M = \int_0^{2\pi} \int_1^2 \left(\frac{1}{r}\right) r dr d\theta = \int_0^{2\pi} \int_1^2 1 dr d\theta = \int_0^{2\pi} 1 d\theta = 2\pi$, so b .

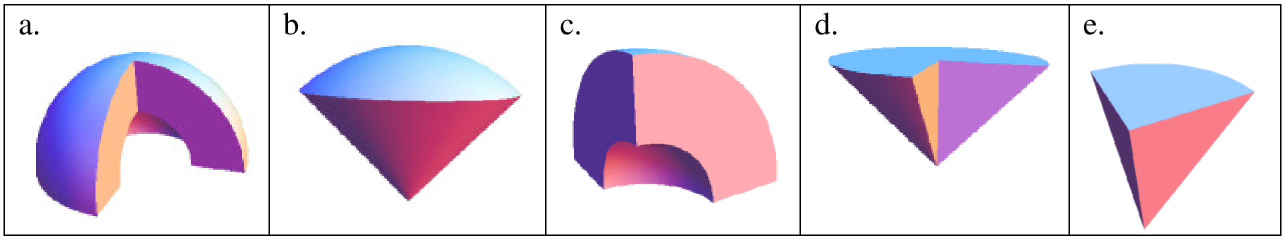
12. Evaluate the triple integral: $\iiint_E x\, dV$ where E is the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $3x + 2y + z = 6$. (Suggestion: set it up and then use your calculator.)

a. 3 b. 4 c. 5 d. 6 e. 7

$\int_0^2 \int_0^{(6-3x)/2} \int_0^{6-3x-2y} x\, dzdydx = 3$ so a.

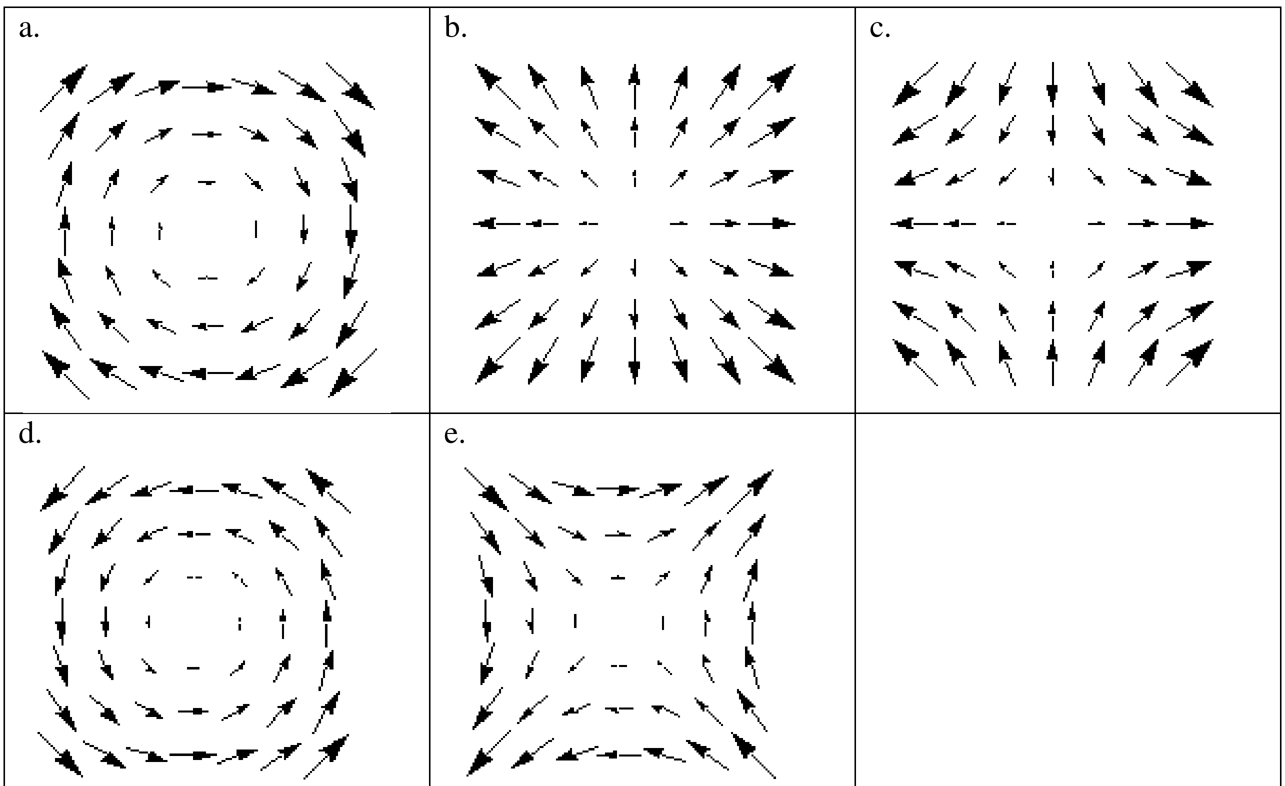
13. Which figure (perhaps rotated) shows the solid described by spherical coordinates

$$0 \leq \rho \leq 2, 0 \leq \phi \leq \pi/4, 0 \leq \theta \leq 2\pi$$



b

14. Match the vector field, $\mathbf{F}(x, y) = \langle y, x \rangle$. (Each picture is in the square $[-1,1] \times [-1,1]$, vector lengths may be scaled.)



For example, $\mathbf{F}(-1,1) = \langle 1, -1 \rangle$ so e .

15. Compute $\int_C y dx + (x + y) dy$ for C the curve given by $x = t + 1, y = 2t, 0 \leq t \leq 1$.

a. 0 b. 5/2 c. 3 d. 7/2 e. 6

Putting everything in terms of *t* we get $\int_0^1 \big((2t)(1) + (t + 1 + 2t)(2)\big) dt = \int_0^1 (8t + 2) dt = 6$ so e .

16. Find $\int_C \nabla f \cdot d\mathbf{r}$, where C is the curve given by $\mathbf{r}(t) = \langle t, t^2 + 1 \rangle$, $0 \le t \le 1$ and f is a function with a continuous gradient and with values $f(x, y)$ given in the table.

$x \setminus y$	0	1	2
0	1	2	3
1	5	8	13
2	19	25	32

a. -4 b. 0 c. 1 d. 4 e. 11

Considering $t = 0$ and $t = 1$ we see that C starts at $(0,1)$ and ends at $(1,2)$. By the Fundamental Theorem of Line Integrals, $\int_C \nabla f \cdot d\mathbf{r} = f(1,2) - f(0,1) = 13 - 2 = 11$. So e .

---

17. Find $\int_C y dx + 4x dy$ where C is a circle described counterclockwise enclosing an area of 4π. (Hint: Green's Theorem – formula given on formula sheet)

a. 2π b. 4π c. 6π d. 12π e. 16π

By Green's Theorem, $\int_C y dx + 4x dy = \iint_D (4-1) dA = 3 area(D) = 12\pi$. d

---

18. For $\mathbf{F}$ a vector field, which of the following expressions is meaningful AND is a vector field?

a. $\text{div}(\text{curl}(\mathbf{F}))$ b. $\text{grad}(\text{curl}(\mathbf{F}))$ c. $\text{curl}(\text{div}(\mathbf{F}))$ d. $\text{curl}(\text{curl}(\mathbf{F}))$ e. $\text{curl}(\text{grad}(\mathbf{F}))$

---

a is meaningful but a scalar, b is meaningless, c is meaningless, d is meaningful and a vector, e is meaningless. So d .

---

19. Which of the following gives a parameterization of a cylindrical surface?

a. $\mathbf{r}(\theta, z) = \langle 2\cos(\theta), 2\sin(\theta), z \rangle$
b. $\mathbf{r}(\theta, z) = \langle z\cos(\theta), z\sin(\theta), z \rangle$
c. $\mathbf{r}(\theta, z) = \langle \theta, z, \theta^2 + z^2 \rangle$
d. $\mathbf{r}(\theta) = \langle 3\cos(\theta), 3\sin(\theta), 3 \rangle$
e. $\mathbf{r}(\theta, z) = \langle \sin(z)\cos(\theta), \sin(z)\sin(\theta), \cos(z) \rangle$

---

a is a cylinder of radius 2. b gives a cone, c a bowl, d a circle, e a sphere.

---

20. Calculate the flux of $\mathbf{F}$ through surface S, in other words, $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and S the surface of the cube with with $0 \le x \le 2$, $0 \le y \le 2$, and $0 \le z \le 2$ and the positive (outward) orientation.

a. 0 b. 3 c. 6 d. 12 e. 24

By the divergence theorem, $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_R \text{div}(\mathbf{F}) dV = \iiint_R 3 dV = 3vol(R) = 24$. e
(Alternatively, the unit normal dot $\mathbf{F}$ is zero on the three faces of the cube meeting the origin and 2 on the other three faces. Each face has area 4, so the flux is $(3)(2)(4) = 24$.)

PART TWO: FREE RESPONSE (50%). The remaining problems are not multiple choice. Answer them on this test paper in the blank space provided. Show the details of your work and box your answers.

21. A projectile is fired with a speed of 500 ft/s at an angle of 45° above horizontal from a hill 300 ft above sea level. Does it pass over a target that is at sea level, 9000 ft away (measured horizontally)? Justify. (Ignore air resistance and use $32\ \text{ft/s}^2$ for the magnitude of acceleration.)

We have $\mathbf{a}(t) = \langle 0, -32\rangle$, $\mathbf{v}(0) = \langle 250\sqrt{2}, 250\sqrt{2}\rangle$, and $\mathbf{r}(0) = \langle 0,300\rangle$. Integrating acceleration gives $\mathbf{v}(t) = \langle c_1, -32t + c_2\rangle$. At $t = 0$ we have $\langle c_1, c_2\rangle = \langle 250\sqrt{2}, 250\sqrt{2}\rangle$ so $\mathbf{v}(t) = \langle 250\sqrt{2}, -32t + 250\sqrt{2}\rangle$. Integrating this gives $\mathbf{r}(t) = \langle 250\sqrt{2}\,t + c_3, -16t^2 + 250\sqrt{2}\,t + c_4\rangle$. At $t = 0$ we have $\langle c_3, c_4\rangle = \langle 0,300\rangle$ so $\mathbf{r}(t) = \langle 250\sqrt{2}\,t, -16t^2 + 250\sqrt{2}\,t + 300\rangle$. The horizontal distance to the target would be reached when $250\sqrt{2}\,t = 9000$ or $t \cong 25.4558$ seconds. The height of the projectile is given by $-16t^2 + 250\sqrt{2}\,t + 300$ which at this time is negative, implying that sea level was struck before reaching the target. So, no, it doesn't pass over the target.

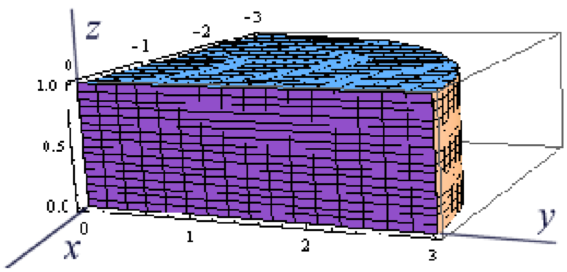
22. A glass aquarium is to be a box without a lid holding a volume of 2 m^3. Find the dimensions that minimize the amount of glass used.

Using dimensions of x, y, z we have that $xyz = 2$ so $z = 2/(xy)$. The amount of glass used is given by $G = xy + 2xz + 2yz = xy + \frac{4}{y} + \frac{4}{x}$. To find the dimensions that minimize $G(x, y)$ we set both partials equal to 0, so $G_x = y - \frac{4}{x^2} = 0$ and $G_y = x - \frac{4}{y^2} = 0$. Hence, substituting $y = \frac{4}{x^2}$ into the last equation gives $x - \frac{x^4}{4} = 0$ or $x(4 - x^3) = 0$. It follows that $\boxed{x = y = \sqrt[3]{4} \cong 1.5874 \text{ and } z = \frac{2}{xy} = 1/\sqrt[3]{2} \cong 0.793701}$. The units are meters. (The minimum amount of glass is $6\sqrt[3]{2} \cong 7.55953$ m^2. Note that $G_{xx} > 0$, $G_{yy} > 0$, and $G_{xy} = 0$ so $D > 0$. From this it follows that we've found a minimum.)

23. a. Describe in words the surface with cylindrical coordinate equation $\theta = \pi/2$.
The *yz*-plane.

b. Sketch the one solid described by all the following cylindrical coordinate inequalities:

$$0 \le r \le 3, \quad \frac{\pi}{2} \le \theta \le \pi, \quad 0 \le z \le 1$$



24. Derive, writing down the appropriate iterated integrals, the volume contained by a sphere of radius a using:

a. Cylindrical coordinates

$$V = \int_0^{2\pi}\int_0^{a}\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}}(1)r\,dzdrd\theta = \int_0^{2\pi}\int_0^{a} 2r\sqrt{a^2-r^2}\,drd\theta = \int_0^{2\pi} -\frac{2}{3}(a^2-r^2)^{\frac{3}{2}}\Big|_{r=0}^{a}\,d\theta = \int_0^{2\pi}\frac{2}{3}a^3\,d\theta = \frac{4}{3}\pi a^3$$

b. Spherical coordinates

$$V = \int_0^{\pi}\int_0^{2\pi}\int_0^{a}(1)\rho^2\sin(\phi)d\rho d\theta d\phi = \int_0^{\pi}\int_0^{2\pi}\frac{1}{3}a^3\sin(\phi)d\theta d\phi = \int_0^{\pi}\frac{2\pi}{3}a^3\sin(\phi)\,d\phi = \frac{4\pi}{3}a^3$$

25. Evaluate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$ where S is the helicoid with vector equation $\mathbf{r}(u,v) = u\cos(v)\,\mathbf{i} + u\sin(v)\mathbf{j} + v\mathbf{k}$ for $0 \le u \le 1, 0 \le v \le \pi$ with upward orientation, and $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z^3\mathbf{k}$. In other words, find the flux of $\mathbf{F}$ across S.

Note that $\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(v) & \sin(v) & 0 \\ -u\sin(v) & u\cos(v) & 1 \end{vmatrix} = \langle \sin(v), -\cos(v), u\rangle$

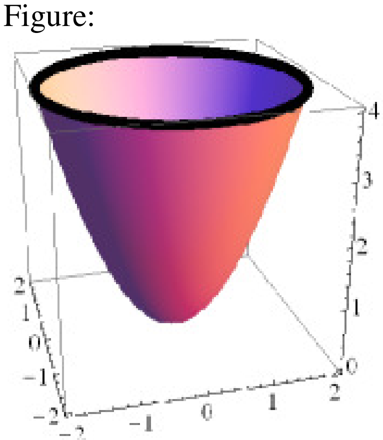
So $\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^{\pi}\int_0^1 \langle u\cos(v), u\sin(v), v^3\rangle \cdot \langle \sin(v), -\cos(v), u\rangle\, dudv =$
$\int_0^{\pi}\int_0^1 (uv^3)\, dudv = \boxed{\frac{\pi^4}{8} \cong 12.176}$.

26. Use Stokes' Theorem to evaluate $\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$ where S is the part of the circular paraboloid (bowl) $z = x^2 + y^2$ below $z = 4$ oriented upward and $\mathbf{F}(x,y,z) = 2yz\mathbf{i} + x^4\mathbf{k}$.

The boundary of S is parameterized by $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), 4\rangle$ for $0 \le t \le 2\pi$ so
$\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \int_{\partial S} \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle (2)(2\sin(t)(4), 0, 16\cos^4(t)\rangle \cdot \langle -2\sin(t), 2\cos(t), 0\rangle dt =$
$\int_{\mathbf{0}}^{\mathbf{2\pi}} -32\sin^2(t)\, dt = \boxed{-32\pi \cong -100.53}$.

(Check: $\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \iint_R \langle 0, 2y - 4x^3, -2z\rangle \cdot \langle -2x, -2y, 1\rangle\, dA =$
$\iint_R 8x^3y - 4y^2 - 2(x^2+y^2)\, dxdy = \iint_R 8x^3y - 6y^2 - 2x^2\, dxdy =$
$\int_0^{2\pi}\int_0^2 (8r^4\cos^3(\theta)\sin(\theta) - 6r^2\sin^2(\theta) - 2r^2\cos^2(\theta))\, r\, dr\, d\theta = -32\pi$)

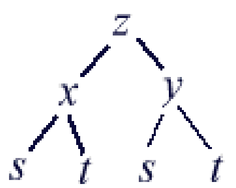
Figure:



CALCULATORS ARE **NOT** PERMITTED FOR PROBLEMS 27, 28, 29, 30.

27. a. Use the Chain Rule (show your work, leave your answer in terms of $x, y, s,$ and t) to find $\partial z/\partial s$ and $\partial z/\partial t$ for

$$z = x^2y^3,\ x = s\cos(t),\ y = e^{2s+t}.$$

	$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = 2xy^3\cos(t) + 3x^2y^2 2e^{2s+t} =$ $2xy^3\cos(t) + 6x^2y^2e^{2s+t}$ $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = 2xy^3 s(-\sin(t)) + 3x^2y^2e^{2s+t} =$ $-2xy^3 s\sin(t) + 3x^2y^2e^{2s+t}$

b. Find the rate of change (the directional derivative) of the function $f(x,y) = x^2y$ at the point (3,4) toward the origin.

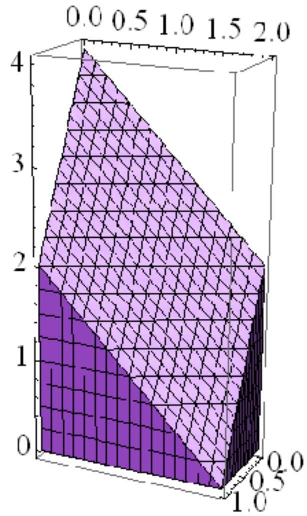
We have $\mathbf{u} = \langle -\frac{3}{5}, -\frac{4}{5}\rangle$ and $\nabla f(x,y) = \langle 2xy, x^2\rangle$, so $\nabla f(3,4) = \langle 24, 9\rangle$.
So $D_{\mathbf{u}}f(3,4) = \langle 24,9\rangle \cdot \langle -\frac{3}{5}, -\frac{4}{5}\rangle = \boxed{-\frac{108}{5} = -21.6}$.

28. a. Calculate the iterated integral: $\int_0^2\int_0^1\int_0^{4-2x-y} dzdxdy$.

$$\int_0^2\int_0^1\int_0^{4-2x-y} dzdxdy = \int_0^2\int_0^1 (4-2x-y)dxdy = \int_0^2 (4x - x^2 - xy)\Big|_{x=0}^{1} dy =$$

$$\int_0^2 (3-y)dy = \left(3y - \frac{1}{2}y^2\right)\Big|_0^2 = \boxed{4}.$$

b. Sketch the solid region of integration for part a.


0.0 0.5 1.0 1.5 2.0
4
3
2
1
0
0.0
0.5
1.0

29. a. Find a function f such that $\mathbf{F} = \nabla f$ for
$\mathbf{F}(x, y, z) = (3yz + 4x)\mathbf{i} + (3xz - 4z)\mathbf{j} + (3xy - 4y)\mathbf{k}$.
Integrating the **i** component with respect to x we get $3xyz + 2x^2$. Similarly, we get antiderivatives $3xyz - 4yz$ and $3xyz - 4yz$. So a common antiderivative is $\boxed{f(x, y, z) = 3xyz + 2x^2 - 4yz}$ which we check has gradient **F**.

b. Use the result in part a to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for C given by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle, 0 \leq t \leq 1$.
The curve C starts at (0,0,0) and ends at $(1,1,1)$. So
$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,1,1) - f(0,0,0) = (3 + 2 - 4) - (0) = \boxed{1}$.

30. Show that the curl of a conservative vector field is zero. More precisely, use the definitions of curl and gradient to prove that for f a function of three variables with continuous second order partial derivatives, $\text{curl}(\nabla f) = \mathbf{0}$.

We have $\text{curl}(\nabla f) = \text{curl}(\langle f_x, f_y, f_z \rangle) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = \langle f_{zy} - f_{yz}, -(f_{zx} - f_{xz}), f_{yx} - f_{xy} \rangle =$
$\langle 0,0,0 \rangle = \mathbf{0}$ where the mixed second order partials are equal by Clairaut's Theorem.