NAME: ________________________________ ALPHA: ____________________

INSTRUCTOR: __________________________ SECTION: _________________

CALCULUS III (SM221) FINAL EXAMINATION

1330 – 1630, 7 May 2012 SHOW ALL WORK IN THIS EXAM PACKAGE

This exam is composed of three parts: Part I - Multiple Choice (50%), Part II – Proof (5%), and Part III – Long Answer (45%). No calculators or notes are allowed for Parts I and II. You must turn in Parts I and II of your exam, including your bubble sheets, before beginning Part III. For Part III you are allowed to use calculators and an 8 ½ x 11 inch formula sheet, written on both sides in your own handwriting. Calculators may not be shared.

PART I: MULTIPLE CHOICE (50%). NO CALCULATORS OR NOTES ALLOWED.
Write your name, alpha number, instructor, and section number on this test and your bubble sheet and bubble in your alpha number. The first 20 problems are multiple-choice. Fill in the best answer to each question on your Scantron bubble sheet. There is no extra penalty for wrong answers on the multiple choice part of the exam.

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1. Which of the following is a parametrization of a cylinder?

a. $\boldsymbol{r}(u,v) = \langle u, v, 3-u-v\rangle$
b. $\boldsymbol{r}(u,v) = \langle u, v, 3u^2+3v^2\rangle$
c. $\boldsymbol{r}(u,v) = \langle 3\cos u, 3\sin u, v\rangle$
d. $\boldsymbol{r}(u,v) = \langle 3v\ \cos u, 3v\sin u, 3v\rangle$
e. $\boldsymbol{r}(t) = \langle 3\cos t, 3\sin t, 0\rangle$

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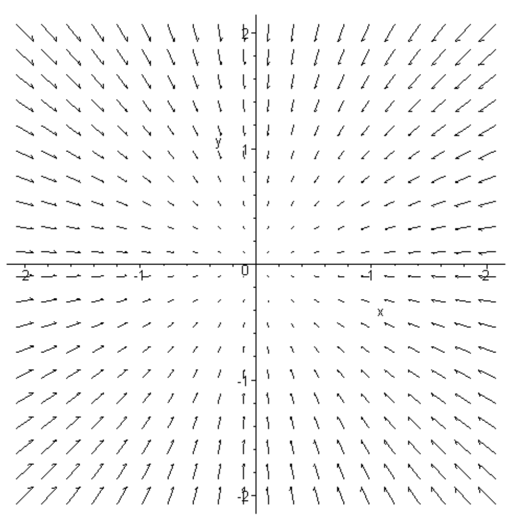
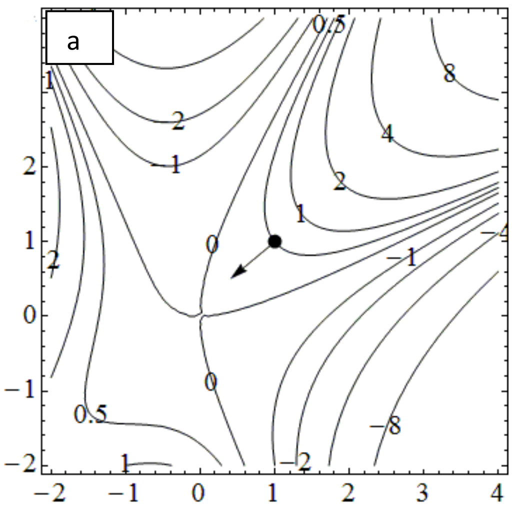
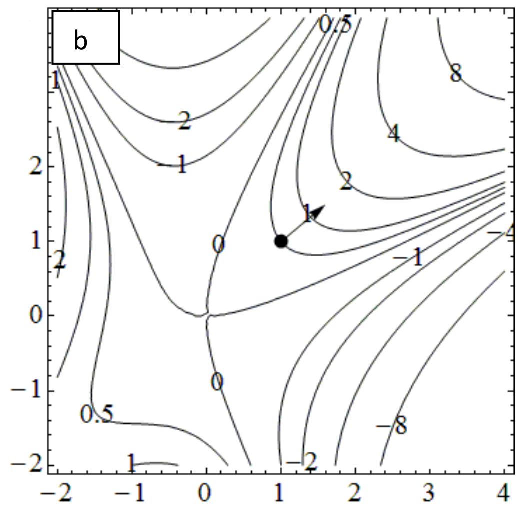
2. Identify the vector field shown below. The lengths of the vectors are not to scale.

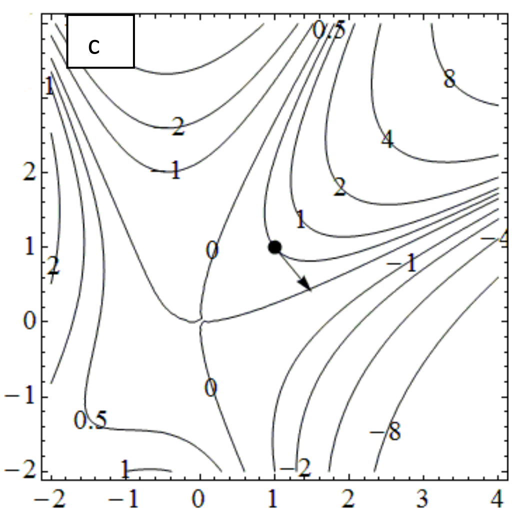
a. $-x\boldsymbol{i} - y\boldsymbol{j}$ b. $y\boldsymbol{i} + x\boldsymbol{j}$ c. $-y\boldsymbol{i} + x\boldsymbol{j}$ d. $\boldsymbol{i} + \boldsymbol{j}$ e. $2\boldsymbol{i}$

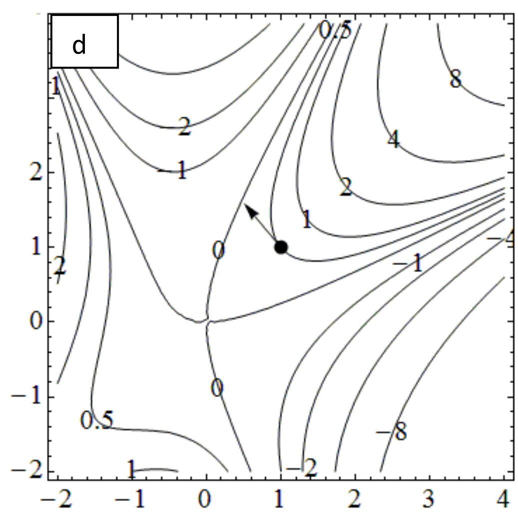


3. Contour curves are shown for a function f. Which graph shows the correct direction for the gradient of f at the point $(1,1)$?









e. The gradient of f at the point $(1,1)$ is the zero vector.

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4. Suppose that the velocity of an object at time t is $\boldsymbol{v}(t) = -2\sin t\,\boldsymbol{i} + 2\cos t\,\boldsymbol{j}$ and the initial position is $\boldsymbol{r}(0) = 5\boldsymbol{i} + 5\boldsymbol{j}$. Which of the following is the position $\boldsymbol{r}(t)$ at time t?

a. $2\cos t\,\boldsymbol{i} + 2\sin t\,\boldsymbol{j}$

b. $(2\cos t + 5)\,\boldsymbol{i} + (2\sin t + 5)\boldsymbol{j}$

c. $(2\cos t + 5t)\,\boldsymbol{i} + (2\sin t + 5t)\boldsymbol{j}$

d. $(2\cos t + 3)\,\boldsymbol{i} + (2\sin t + 3)\boldsymbol{j}$

e. $(2\cos t + 3)\,\boldsymbol{i} + (2\sin t + 5)\boldsymbol{j}$

5. The volume of a cylinder of radius r and height h is given by $V = \pi r^2 h$, where r and h are measured in centimeters and t in seconds. Suppose that at a certain time the radius is 10 cm, the height is 12 cm, the radius is increasing at a rate of 2 cm/sec, and the height is decreasing at a rate of 1 cm/sec. At what rate is the volume changing at this moment?

a. decreasing at 40π cm^3/sec

b. increasing at 380π cm^3/sec

c. increasing at 580π cm^3/sec

d. increasing at 1200π cm^3/sec

e. increasing at2400π cm^3/sec

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6. The following table shows the depth of water in a swimming pool measured in feet. Use the Midpoint Rule, with $m = 2$ subdivisions for x and $n = 2$ subdivisions for y to estimate the volume of water in the pool. (The x-scale for the table is given in the leftmost column and the y-scale is given in the top row.)

x \ y	0	5	10	15	20
0	9	8	6	4	2
10	13	10	7	5	3
20	15	12	11	5	5
30	14	11	9	4	4
40	12	10	8	4	3

a. 300 cubic feet

b. 3000 cubic feet

c. 6000 cubic feet

d. 8000 cubic feet

e. 24,000 cubic feet

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7. Use the chart in the previous problem. Let z be the depth in feet as a function of x and y. Which of the following gives the best approximation of $\frac{\partial z}{\partial y}$ at the point at which $x = 10$ and $y = 5$? (The units for the answers listed below are in feet of depth per horizontal foot.)

a. $-\frac{10}{6}$ b. $-\frac{6}{10}$ c. -6 d. $\frac{20}{4}$ e. 0

8. Suppose that a hill has the shape

$$z = 100 - .01x^2 - .03y^2$$

where x, y, and z are measured in meters, the positive x-axis points east, and the positive y-axis points north. If you are standing at the point at which $x = 1$ and $y = 1$ and you are walking east, are you ascending or descending and at what rate?

a. ascending at .01 m per horizontal m

b. ascending at .03 m per horizontal m

c. descending at .02 m per horizontal m

d. descending at .04 m per horizontal m

e. neither ascending nor descending

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9. Converting the integral

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} x^2 + y^2\, dy\, dx$$

to polar coordinates gives which of the following integrals?

a. $\int_0^{\pi}\int_0^2 r^2\, dr\, d\theta$

b. $\int_0^{2\pi}\int_0^2 r^2\, dr\, d\theta$

c. $\int_0^{\pi}\int_0^2 r^3\, dr\, d\theta$

d. $\int_0^{2\pi}\int_0^2 r^3\, dr\, d\theta$

e. $\int_{-2}^{2}\int_0^{\sqrt{4-r^2\cos^2\theta}} r^2\, dr\, d\theta$

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10. The integral

$$\int_0^2\int_{3x}^{6} \sin(y^2) dy\, dx$$

is equal to which of the following?

a. $\int_0^6\int_0^{\frac{y}{3}} \sin(y^2) dx\, dy$

b. $\int_0^6\int_{\frac{y}{3}}^{2} \sin(y^2) dx\, dy$

c. $\int_0^2\int_{\frac{y}{3}}^{6} \sin(y^2) dx\, dy$

d. $\int_{3x}^6\int_0^2 \sin(y^2) dx\, dy$

e. $\int_0^6\int_0^2 \sin(y^2) dx\, dy$

11. Which of the following integrals gives the volume of the region in the first octant bounded by the planes $4x + 2y + z = 4$, $x = 0$, $y = 0$, and $z = 0$?

a. $\int_0^1 \int_0^2 \int_0^4 4 - 4x - 2y\, dz\, dy\, dx$

b. $\int_0^1 \int_0^2 \int_0^{4-4x-2y} 1\, dz\, dy\, dx$

c. $\int_0^1 \int_0^{2-2x} \int_0^{4-4x-2y} 4 - 4x - 2y\, dz\, dy\, dx$

d. $\int_0^1 \int_0^{2-2x} \int_0^{4-4x-2y} 1\, dz\, dy\, dx$

e. $\int_0^1 \int_0^2 \int_0^4 1\, dz\, dy\, dx$

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12. Which of the following expressions gives the integral of $f(x, y, z) = x^2 + y^2 + z^2$ over the region inside the cylinder $x^2 + y^2 = 1$ between $z = 0$ and $z = 2$?

a. $\int_0^{2\pi} \int_0^1 \int_0^2 (r^2 + z^2) r\, dz\, dr\, d\theta$

b. $\int_0^{2\pi} \int_0^1 \int_0^2 (1 + z^2) r\, dz\, dr\, d\theta$

c. $\int_0^{2\pi} \int_0^1 \int_0^2 (r^2 + 4) r\, dz\, dr\, d\theta$

d. $\int_0^{2\pi} \int_0^1 \int_0^{\sqrt{1-r^2}} r^3\, dz\, dr\, d\theta$

e. $\int_0^{2\pi} \int_0^1 \int_0^2 1\, dz\, dr\, d\theta$

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13. Which of the following integrals gives the volume of the region which is inside the sphere $x^2 + y^2 + z^2 = 4$ and above the cone $z = \sqrt{x^2 + y^2}$?

a. $\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^1 \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$

b. $\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^2 \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$

c. $\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^2 \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$

d. $\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^4 \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$

e. $\int_0^{2\pi} \int_0^{\pi} \int_0^1 1\, d\rho\, d\varphi\, d\theta$

14. A lamina has the shape of the rectangle $0 \leq x \leq 3,\ 0 \leq y \leq 2$, where x and y are measured in centimeters. The density at any point (x, y) in the lamina is $\rho(x, y) = 2x$ grams per square centimeter. Which of the following is the mass of the lamina?

a. 6 grams
b. 18 grams
c. 24 grams
d. 36 grams
e. 54 grams

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15. Suppose that $f(x, y, z) = 3x^2 + 3y^2z + y^3$. Which of the following is equal to $\nabla f(x, y, z)$?

a. $6x\boldsymbol{i} + (6yz + 3y^2)\boldsymbol{j} + 3y^2\boldsymbol{k}$
b. $6x\boldsymbol{i} + 6yz\boldsymbol{j} + 0\boldsymbol{k}$
c. $6x + 6yz + 6y^2$
d. $x^3 + y^3z$
e. 0

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16. Evaluate the line integral $\int_C 3y^2dx + x^3dy$ where C is the curve given by the parametrization $\boldsymbol{r}(t) = \langle t, t^2 \rangle$ for $0 \leq t \leq 1$.

a. 1
b. $5/4$
c. $27/20$
d. -3
e. 0

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17. Suppose that $f(x, y) = \frac{1}{\sqrt{x^2+y^2}}$ and $\boldsymbol{F} = \nabla f$. Use the fundamental theorem of line integrals to calculate the work done by $\boldsymbol{F}$ along a line segment C from the point $(3,4)$ to the point $(1,0)$.

a. -4
b. $\sqrt{20}$
c. $1/5$
d. $4/5$
e. 0

18. Suppose that D is the square $-1 \le x \le 1, -1 \le y \le 1$ and C is the boundary of D with the counterclockwise orientation. Which of the following is equal to

$$\int_C (y+1)dx + (2y-2)dy?$$

a. 2 b. 0 c. -1 d. -4 e. $\boldsymbol{i} - 2\boldsymbol{j}$

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19. Suppose that S is the half of the sphere $x^2 + y^2 + z^2 = 4$ such that $z \ge 0$, with the upward orientation (i.e., in the direction of the positive z-axis) and C is the boundary of S with the positive orientation. Given $\boldsymbol{F} = \langle z - y, x, 0\rangle$ and $\text{curl } \boldsymbol{F} = \langle 0,1,2\rangle$ find the flux of curl $\boldsymbol{F}$ across S, i.e., find

$$\iint_S \text{curl } \boldsymbol{F} \cdot d\vec{\boldsymbol{S}}.$$

a. 16π b. 8π c. 4π d. 0 e. $2\boldsymbol{k}$

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20. Suppose that S is the surface of the cube $0 \le x \le 2, 0 \le y \le 2, 0 \le z \le 2$ and $\boldsymbol{F}(x,y,z) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$. Which of the following is equal to the flux of $\boldsymbol{F}$ across S, i.e., which is equal to

$$\iint_S \boldsymbol{F} \cdot d\vec{\boldsymbol{S}}?$$

a. $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ b. 24 c. 8 d. 3 e. 0

PART II: PROOF (5%). NO CALCULATORS OR NOTES ALLOWED.
You must turn in Parts I and II of your exam, including your bubble sheets, before beginning Part III

21. Write ONE of the following two proofs.
Prove: If f is a differentiable function of two or three variables and $\boldsymbol{u}$ is a unit vector, the maximum value of the directional derivative of f in the direction of $\boldsymbol{u}$ is $|\nabla f(x)|$ and it occurs when $\boldsymbol{u}$ is in the direction of $\nabla f(x)$.

OR

Prove: If $\boldsymbol{F} = P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k}$ is a conservative vector field whose components have continuous first-order derivatives, then curl $\boldsymbol{F} = \boldsymbol{0}$, where

$$\text{curl } \boldsymbol{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\boldsymbol{k}.$$

PART III: LONG ANSWER (45%). CALCULATORS AND FORMULA SHEETS PERMITTED.
You must turn in Parts I and II of your exam, including your bubble sheets, before beginning Part III. Calculators are permitted for this part of the exam. Calculators may not be shared. One 8 ½ x 11 inch formula sheet, in your own handwriting, written on both sides, is permitted for this part of the exam. Show details of your work and box your answers.

22. Find all local maxima, local minima, and saddle points of $f(x, y) = x^3 - 3xy + y^3$. Be sure to show how the second derivative test was used.

23. a. Sketch the region bounded by the surface $z = 1 + x^2 + y^2$ and the plane $z = 5$.

b. Find the volume of the region.

24. a. Given $\boldsymbol{F}(x, y) = 3y\boldsymbol{i} + (3x + 4y)\boldsymbol{j}$ find a function f such that $\boldsymbol{F} = \nabla \text{f}$.

b. Let C be the line segment from the point $(1,0)$ to the point $(0,2)$. Use your answer from part (a) to evaluate the following integral without parametrizing C.

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$$

c. Check your answer to the previous problem by parametrizing the line segment C from the point $(1,0)$ to the point $(0,2)$ and calculating the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ directly.

25. Suppose that the temperature at a point (x, y, z) is given by

$$T(x, y, z) = \frac{30}{1+x^2+y^2+z^2}$$

where T is measured in °C and x, y, and z in meters. Find the rate of change of temperature at the point $(1,1,1)$ in the direction of the vector $\langle 2, -2, 1 \rangle$. Explain in words what your answer means.

26. Suppose that $\boldsymbol{F}(x, y) = 2y\boldsymbol{i} - 2x\boldsymbol{j}$ and C is the circle $x^2 + y^2 = 25$, with the counterclockwise direction.

a. Sketch the vector field $\boldsymbol{F}$ and the curve C on the same graph. Be sure that your sketch shows the direction of $\boldsymbol{F}$ at at least four points of C.

b. Use Green's Theorem to calculate $\int_C 2y\,dx - 2x\,dy$.

c. Is $\boldsymbol{F}$ a conservative vector field? Explain.

27. A thin wire has the shape of a helix, with parametrization $x = 2\cos t$, $y = 2\sin t$, and $z = t$, for $0 \le t \le \pi$. Suppose that the density at any point (x, y, z) on the wire is given by $\rho(x, y, z) = 0.1\sqrt{1 + z^2}$. Find the mass of the wire, i.e., find the integral of $0.1\sqrt{1 + z^2}$ over the given curve. (You may assume that x, y, and z are measured in centimeters and the density is measured in grams per centimeter.)

28. A solid is bounded by the plane $z = 0$ and the top half of the sphere $x^2 + y^2 + z^2 = 9$. Suppose that the density at any point in the solid is twice the distance from the origin. Find the mass of the solid. (You may assume that x, y, and z are measured in centimeters and the density is measured in grams per cubic centimeter.)

29. Suppose that $\boldsymbol{F}(x, y, z) = y\boldsymbol{i} + 2x\boldsymbol{j} + z\boldsymbol{k}$. Find the flux of $\boldsymbol{F}$ upward through the part of the plane $x + y + z = 3$ in the first octant.

30. Gauss' Law of electrostatics states that the net charge Q enclosed by a closed surface S is given by a constant ε_0 times the flux of the electric field $\boldsymbol{E}$, i.e.,

$$Q = \varepsilon_0 \iint_S \boldsymbol{E} \cdot d\boldsymbol{S}.$$

Suppose that $\boldsymbol{E}(x, y, z) = 2x\boldsymbol{i} + y\boldsymbol{j} + 3z\boldsymbol{k}$ and S is the surface of the cube $0 \le x \le 2$, $0 \le y \le 2$, $0 \le z \le 2$. Find the charge enclosed by S. (Hint: Use one of the theorems studied in this course.)