The exam has three parts: Part 1: 15 multiple choice problems: 10 points per problem; Part 2: 3 long answer problems, 30 to 40 points per problem; Part 3: one Fourier series problem allowing the use of calculators, 30 points (plus a bonus problem). At the end of Part 3 you can find formulas and one blank page.

NO CALCULATORS ALLOWED FOR PARTS 1 and 2 OF THE EXAM. FORMULAS ARE ALLOWED FOR THE WHOLE EXAM. Most problems, including the multiple choice, require work before identifying the right answer. Show the work supporting your answer.

YOU MUST TURN IN YOUR BUBBLE SHEET and PARTS 1 and 2 BEFORE TAKING OUT YOUR CALCULATOR.

PART 1: Problems 1- 15, 10 points each. No Calculators; show supporting computations/work.

1. For which value of m is the function $y = t^m$ a solution of the differential equation $t^2 y'' + \frac{1}{4} y = 0$?

a) $m = 1$ b) $m = 1/2$ c) $m = 1/4$ d)$m = -1/4$ e) $m = -1/2$.

2. The linear ODE $y^{(3)} - y'' + y' + y = 0$ has the fundamental solutions $y = e^t,\ y = \sin t$, and $y = \cos t$. The solution to the ODE with initial values $y(0) = 0,\ y'(0) = 0,\ y''(0) = 4$ is $y =$

a) $2 - 2\cos t$ b) $2e^t - 2\cos t$ c) $2e^t - 2\sin t - 2\cos t$ d) $2\sin t + 2\cos t - 2$ e) $4e^t + 2\sin t - 4\cos t$

3. A tank contains 70 gallons of water with 10 lb of salt dissolved in it. Brine with concentration of 2 lb/gal is pumped into the tank at a rate of 2 gal/min. The well-stirred mixture is pumped out at the rate of 4 gal/min. A differential equation for the amount A (in lb) of salt in the tank is

a) $\frac{dA}{dt} = 10 - \frac{2}{35} A$ b) $\frac{dA}{dt} = 4 - \frac{2}{35} A$ c) $\frac{dA}{dt} = 10 - \frac{4}{70-2t} A$ d) $\frac{dA}{dt} = 4 - \frac{2}{35-t} A$

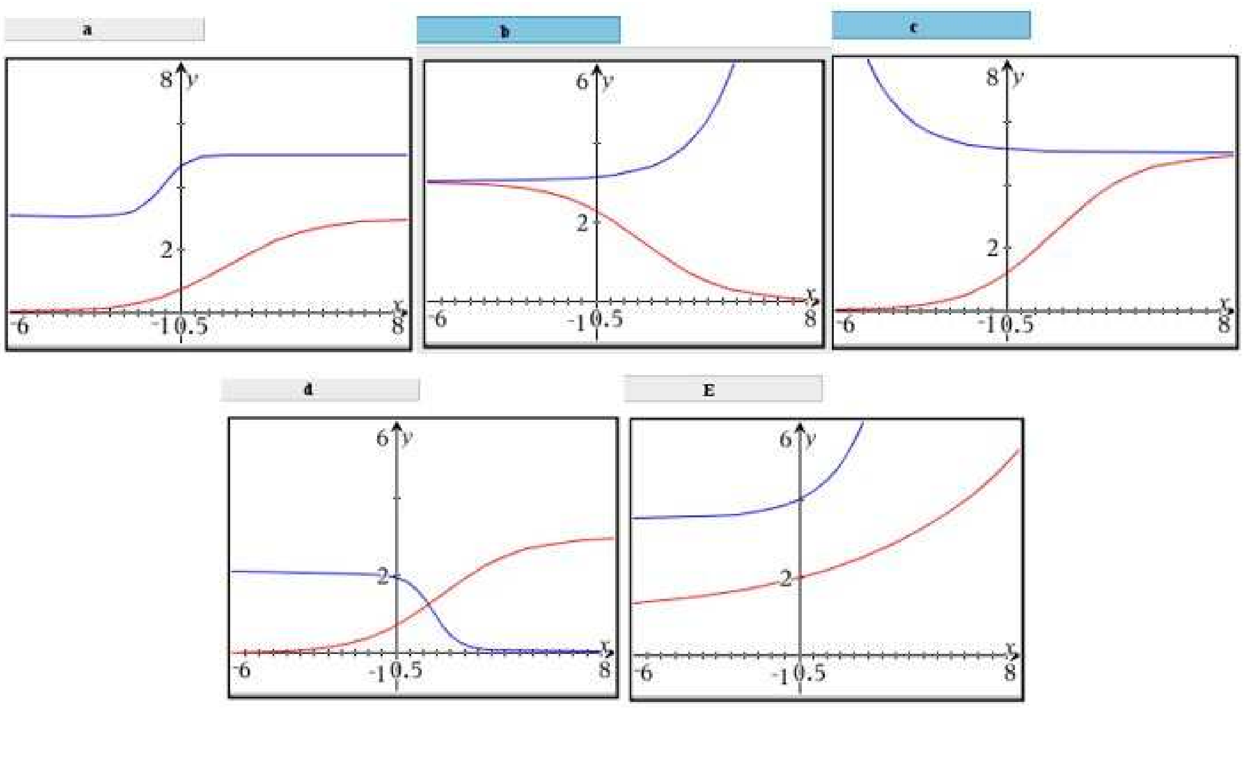
4. The general solution for the separable equation $\frac{1}{y}\frac{dy}{dt} - \cos t = 4$ is:

a) $y = Ce^{4t+\sin t}$ b) $y = \sin t + e^{4t} + C$ c) $y = (4t + \sin t)y + C$ d) $y = 4t + C\sin t$ e) $y = A\sin t + B\cos t$.

5. Which of the following functions is the general solution to the linear ODE $\frac{dy}{dt} + \frac{1}{t}y = 4$?

a) $y = 4t + y\ln t$ b) $y = 4t + C$ c) $y = 2t + C$ d) $y = 2t + \frac{C}{t}$ e) $y = e^{4t+\ln t+C}$

6. The differential equation $\frac{dy}{dx} = 3y(5-y)$ models some population growth. Without solving it, find the equilibrium (constant) solutions and the phase portrait. Use those to identify which of the following graphs captures two solution curves:



7. Given that $y = t^3$ and $y = t^5$ are linearly independent solutions of the homogeneous equation

$$t^2 y'' - 7ty' + 15y = 0,$$

and that $y = \ln t + 0.125t$ is a particular solution of the non-homogeneous equation

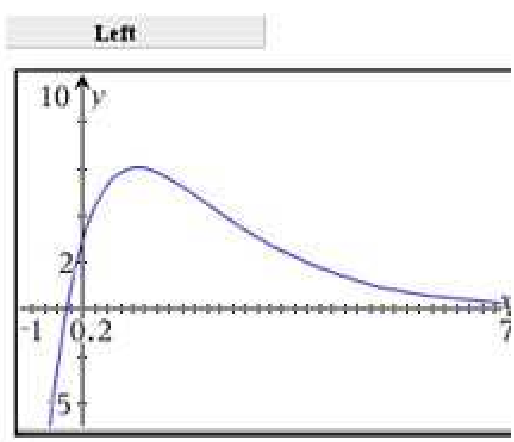
$$t^2 y'' - 7ty' + 15y = 15 \ln t - 7$$

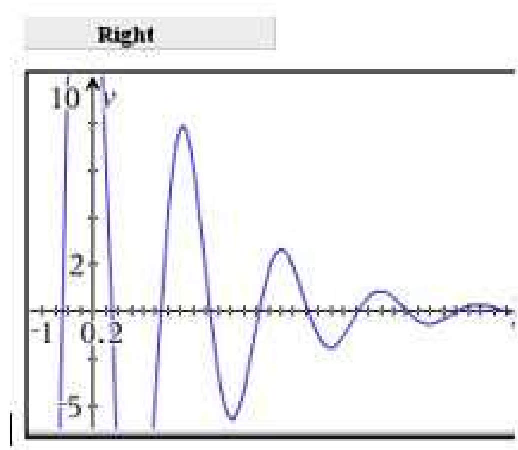
which of the following is the general solution of the non-homogenous equation?

a) $y = c_1 \ln t + c_2(0.125t) + c_3 t^3 + c_4 t^5$ b) $y = c_1(\ln t + 0.125t) + c_2 t^3 + c_3 t^5$

c) $y = \ln t + 0.125t + c_1 t^3 + c_2 t^5$ d) $y = c_1 t^3 + c_2 t^5$ e) $y = c_1 \ln t + c_2(0.125t)$.

8. The deformation y of a damped spring is described by $y'' + 6y' + ky = 0$, where k is the spring constant. The spring motion is OVER-damped. Select the graph of y and the value of k from the options below that are consistent with OVER-damping.





a) Left graph, $k = 25$ b) Left graph, $k = 8$ c) None of the graphs works

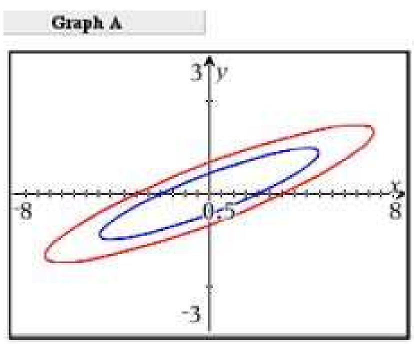
d) Right graph, $k = 25$ e) Right graph, $k = 8$.

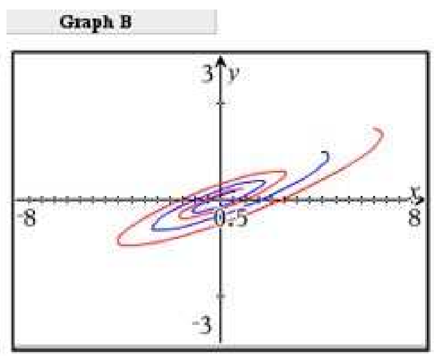
9. A simple circuit consists of an inductance of 4 henries, a capacitance of 0.01 farads, and a voltage source that supplies E(t) volts. For which of the following functions E(t) is the charge q of the system in resonance?

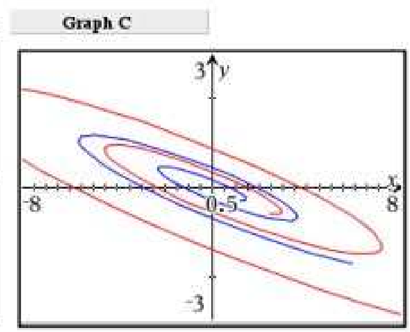
a) $100\cos t$ b) $100\cos(5t)$ c) $100\sin(10t)$ d) $25\cos(0.2t)$ e) $25\sin(0.1t)$.

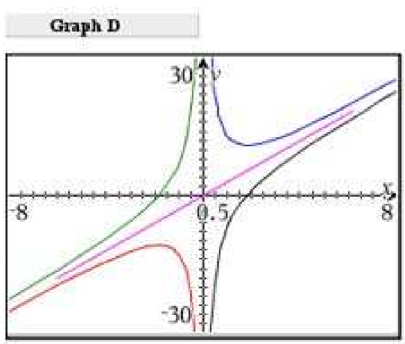
10. Which of the following is the phase portrait for the system $\frac{dx}{dt} = -x + 5y$, $\frac{dy}{dt} = -x + y$?

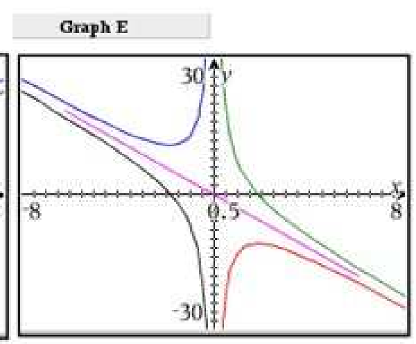
a) Graph A b) Graph B c) Graph C d) Graph D e) Graph E











11. A battle with $x(t)$ troops on one side and $y(t)$ on the other is modeled by $x' = -4y + 8,\ y' = -x + 5$, or in vector notation, by

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}, \quad \text{where } A = \begin{bmatrix} 0 & -4 \\ -1 & 0 \end{bmatrix},\ \vec{f} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

The eigenvalues of A are $2, -2$ with corresponding eigenvectors $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Which of the following gives the general solution $\vec{x}$ for the forced system?

a) $\begin{bmatrix} 8 \\ 5 \end{bmatrix} + c_1 e^{2t} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$;

b) $\begin{bmatrix} 5 \\ 2 \end{bmatrix} + c_1 e^{2t} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$;

c) $c_1 e^{2t} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$;

d) $\begin{bmatrix} 8 \\ 5 \end{bmatrix} + c_1 \cos(2t) \begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2 \sin(2t) \begin{bmatrix} 2 \\ 1 \end{bmatrix}$;

e) $\begin{bmatrix} 5 \\ 2 \end{bmatrix} + c_1 \cos(2t) \begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2 \sin(2t) \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

12. Consider the system $\frac{dx}{dt} = 8 - y^3,\ \frac{dy}{dt} = x + y - 7$. Which of the following is an equilibrium solution?

a) $(0, 0)$ b) $(0, 2)$ c) $(5, 0)$ d) $(5, 2)$ e) multiple critical points: all of those listed in a-d.

13. Consider the function g given on its half period by $g(x) = \begin{cases} 0 & \text{for } 0 < x \leq 1/2, \\ 5 & \text{for } 1/2 < x < 1 \end{cases}$, and answer the following about its half-range cosine expansion. The coeficient of $\cos(5\pi x)$ is

a) $\frac{-10}{\pi}$ b) $\frac{-2}{\pi}$ c) 0 d) $\frac{-2}{\pi}$ e) 10π

14. Consider the function f with period 2π , given over one period by $f(x) = \begin{cases} -x^2 & -\pi < x < 0 \\ x^2 & 0 < x < \pi \end{cases}$
Graph the function over two periods and use the graph to answer the following questions about its Fourier series:

a) $a_n = 0$ for $n = 0, 1, 2, ...$
b) $a_n = 0$ for $n = 1, 2, 3, ...$, but $a_0 \neq 0$.
c) $b_n = 0$ for $n = 1, 2, 3, ...$
d) There are non-zero terms among a_n and there are non-zero terms among b_n for $n = 0, 1, 2, ...$
e) No Fourier series exist for discontinuous functions.

15. If $u(x, y) = X(x)Y(y)$ is a product solution of the differential equation

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y},$$

then one can separate variables to obtain that there is a constant λ so that

a) $X'' = \lambda X$; b) $X'' - X' = \lambda X$; c) $X'' - X = \lambda X'$;

d) $X'' = \lambda(X + X')$; e) $X'' = \lambda(X - X')$;

Part II, Problem 1. 30 points. Choose ONE of the following two options:

Option A: Consider the differential equation

$$D^2(D^2 - 4D + 5)(D - 5)(D^2 + 9)^2[y(t)] = 4 + te^{5t} + 8\sin(3t) + te^t\cos(3t)$$

(a) Find the form of the general homogeneous solution.

(b) Without determining any coefficients, determine the form for the particular solution.

Option B: Use undetermined coefficients to solve the initial value problem

$$y'' + 6y' + 8y = 12e^{-2x}, y(0) = -1, y'(0) = 2.$$

Solution to Option ____

Part II, Problem 2. 40 points. The nonlinear system

$$\frac{dx}{dt} = xy - x$$

$$\frac{dy}{dt} = y - \frac{1}{2}y^2 - \frac{1}{2}xy$$

has $M(1,1)$ and $N(0,2)$ among its critical points. These are the only critical points in the half plane $y > 0$.

a. Linearize the system near M and sketch the phase portrait of the linear system. What does the phase portrait tell about M? Circle one: M is a: spiral, node, saddle, neither.

b. Linearize the system near N and sketch the phase portrait of the linear system. What does the phase portrait tell about N? Circle one: N is a: spiral, node, saddle, neither.

c. Use the linearizations from parts a) and b) to sketch the phase portrait for the system in the region $y > 0$. You don't have to concern yourselves with $y \leq 0$.

Part II, Problem 3. 40 points. Consider the heat equation

$$\frac{\partial u}{\partial t} = 0.1\frac{\partial^2 u}{\partial x^2} \quad \text{for } 0 < x < 1, \quad t > 0$$

with the zero ends condition $u(0,t) = u(1,t) = 0$.

Find the general solution using separation of variables. Show your work; you don't have to include the case of non-working exponentials. Use summation notation in your final answer.

Problem 1: Let f be the function of period 6 such that

$$f(x) = \begin{cases} 0 & \text{for } -3 < x \leq 0, \\ x & \text{for } 0 < x < 3 \end{cases}$$

a) Sketch the graph of f on the interval [-6, 6]. At which points in this interval is f discontinuous?

(b) Find the Fourier series of f.

(c) To what value does the Fourier series converge at x = 0? At x = 3?

Bonus: Find the general solution and a fundamental matrix Φ for the system

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \vec{x}.$$