SM 223 Calculus 3 FINAL EXAMINATION 07 December 2007-08 1330-1630
Part I Multiple Choice NO CALCULATOR ALLOWED

SOLUTIONS

1. $\boxed{\text{d}}$ $|\mathbf{a}-\mathbf{b}| = |\,(\mathbf{i}+2\mathbf{k})-(\mathbf{i}-2\mathbf{j})\,| = |2\mathbf{j}+2\mathbf{k}| = \sqrt{0^2+2^2+2^2} = \sqrt{8}$.

2. $\boxed{\text{d}}$ **Method 1:** $|\overrightarrow{PQ}\times\overrightarrow{PS}|$ = area of parallelogram $PQRS$ = base times height = $(4)(4) = 16$
Method 2: $|\overrightarrow{PQ}\times\overrightarrow{PS}| = |\overrightarrow{PQ}||\overrightarrow{PS}|\sin(\theta) = (4)(4\sqrt{2})\sin(45°) = (4)(4\sqrt{2})\frac{\sqrt{2}}{2} = 16$.
Method 3: In a suitable coordinate system $|\overrightarrow{PQ}\times\overrightarrow{PS}| = |4\mathbf{i}\times(4\mathbf{i}+4\mathbf{j})\,| = |16\mathbf{k}| = 16$.
The cross product could be computed either with the determinant $\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & 0 & 0\\ 4 & 4 & 0\end{vmatrix} = 16\mathbf{k}$, or by appealing to distributivity: $4\mathbf{i}\times(4\mathbf{i}+4\mathbf{j}) = (4\mathbf{i}\times 4\mathbf{i}) + (4\mathbf{i}\times 4\mathbf{j}) = \mathbf{0}+16\mathbf{k}$.

3. $\boxed{\text{e}}$ Since $r^2 = x^2+y^2$, the given equation becomes $x^2+y^2+z^2 = 100$ in Cartesian coordinates. This is a sphere with radius 10.

4. $\boxed{\text{a}}$ Note that $\mathbf{r}(0) = \langle 1,0\rangle$. Thus we seek $|\mathbf{r}'(0)|$. Now $\mathbf{r}'(t) = \langle 2e^{2t}, e^t\rangle$ and $\mathbf{r}'(0) = \langle 2,1\rangle$. So $|\mathbf{r}'(0)| = |\langle 2,1\rangle| = \sqrt{2^2+1^2} = \sqrt{5}$.

5. $\boxed{\text{e}}$ $\mathbf{r}(t) = \int \mathbf{r}'(t)\,dt = \int\langle 2t, 3t^2\rangle\,dt = \langle t^2, t^3\rangle + \mathbf{C}$. Also, $\langle 1,4\rangle = \mathbf{r}(1) = \langle 1^2, 1^3\rangle + \mathbf{C}$, and so $\langle 0,3\rangle = \mathbf{C}$. Therefore $\mathbf{r}(t) = \langle t^2,t^3\rangle + \langle 0,3\rangle$, and $\mathbf{r}(2) = \langle 2^2,2^3\rangle + \langle 0,3\rangle = \langle 4,11\rangle$.

6. $\boxed{\text{d}}$ The gradient vector must be perpendicular to the level curve through the origin, which eliminates all the given vectors except $\mathbf{j}$ and $-\mathbf{j}$. Also, the gradient must point in the direction of maximum rate of increase of f at the origin. From the labels on the level curves the function f increases in the direction of the negative y-axis at the origin, so the only choice is $-\mathbf{j}$.

7. $\boxed{\text{e}}$ We have $F_x(x,y) = 3x^2y^2$ and $F_{xy} = 6x^2y$, and so $F_{xy}(2,3) = 6\cdot 2^2\cdot 3 = 72$.

8. $\boxed{\text{a}}$ $\nabla f(0,0) = \langle f_x(0,0), f_y(0,0)\rangle = \langle e^x\sin(y)+2(x-1),\ e^x\cos(y)\rangle\Big|_{(x,y)=(0,0)} = \langle -2,1\rangle$. So the directional derivative is $D_{\mathbf{u}}\left(f(0,0)\right) = \nabla f(0,0)\cdot\mathbf{u} = \langle -2,1\rangle\cdot\langle\frac{3}{5},\frac{4}{5}\rangle = -\frac{6}{5}+\frac{4}{5} = -\frac{2}{5}$.

9. $\boxed{\text{c}}$ The points (x,y) in the region of integration satisfy $0\le y\le 4$ and $\sqrt{y}\le x\le 2$. This region lies below the parabola $y = x^2$ and above the x-axis for $0\le x\le 2$. It may also be described as $0\le x\le 2$ and $0\le y\le x^2$

10. $\boxed{\text{e}}$ $\int\int_R\ 12(x^2+y^2)\,dA = \int_0^{2\pi}\int_1^2 12r^2\ r\,dr\,d\theta = \int_0^{2\pi}\int_1^2 12r^3\,dr\,d\theta = \int_0^{2\pi} 3r^4\big|_{r=1}^{r=2}\,d\theta$
$= \int_0^{2\pi}(3\cdot 2^4 - 3\cdot 1^4)\ d\theta = 45\int_0^{2\pi}\ d\theta = 90\pi$.

11. $\boxed{\text{d}}$ $\cos(\theta) = \frac{\langle 2,-2,1\rangle\cdot\langle 2,1,2\rangle}{|\langle 2,-2,1\rangle||\langle 2,1,2\rangle|} = \frac{4}{(3)(3)} = \frac{4}{9}$

12. $\boxed{\text{e}}$ The direction vector of the given line is $\mathbf{v} = \langle 1,2,3\rangle$. The normal vector to the plane in (e) is $\mathbf{N}_e = \langle 3,-3,1\rangle$. Since $\mathbf{v}\cdot\mathbf{N}_e = 3-6+3 = 0$, we know that $\mathbf{v}$ and $\mathbf{N}_e$ are perpendicular, and hence the plane in (e) is parallel to the given line. None of the other planes has this property.

13. $\boxed{\text{a}}$ The velocity is $\mathbf{r}'(t) = \langle -4\sin(t), 4\cos(t), 3\rangle$. So the speed is $|\mathbf{r}'(t)| = |\langle -4\sin(t), 4\cos(t), 3\rangle| = \sqrt{(-4\sin(t))^2+(4\cos(t))^2+(3)^2} = \sqrt{16\sin^2(t)+16\cos^2(t)+9} = \sqrt{16\left(\sin^2(t)+\cos^2(t)\right)+9} = \sqrt{16+9} = 5$. Therefore the arclength is $L = \int_0^\pi|\mathbf{r}'(t)|\,dt = \int_0^\pi 5\,dt$.

SOLUTIONS

14. $\boxed{\text{a}}$ By the Chain Rule $\frac{dA}{dt} = \frac{\partial A}{\partial x}\frac{dx}{dt} + \frac{\partial A}{\partial y}\frac{dy}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt} = 6(-3) + 5(4) = 2.$

15. $\boxed{\text{b}}$ The linear approximation of f near $(1,2)$ is

$$\begin{array}{rcccccc} f(x,y) & \approx & f(1,2) & + & f_x(1,2)(x-1) & + & f_y(1,2)(y-2) \\ & = & 5 & + & 3(x-1) & + & 4(y-2). \\ \text{So } f(1.1,1.9) & \approx & 5 & + & 3(0.1) & + & 4(-0.1) \\ & = & 4.9. & & & & \end{array}$$

16. $\boxed{\text{a}}$ The plane is of the form $A(x-1) + B(y-2) + C(z-0) = 0$. The normal vector to the level surface $F(x,y,z) = x^2 + y^2 + xe^z = 6$ is

$$\langle A, B, C\rangle = \nabla F(1,2,0) = \langle 2x + e^z, 2y, xe^z\rangle\Big|_{(x,y,z)=(1,2,0)} = \langle 3, 4, 1\rangle.$$

So the plane is $3(x-1) + 4(y-2) + 1(z-0) = 0$, which may be re-written as choice (a).

17. $\boxed{\text{b}}$

$$\left\{\begin{array}{c} F_x = 0 \\ F_y = 0 \end{array}\right\} \qquad \left\{\begin{array}{c} 2x = 0 \\ 2y - 6 = 0 \end{array}\right\} \qquad \left\{\begin{array}{c} x = 0 \\ y = 3 \end{array}\right\}$$

So $(x,y) = (0,3)$ is the only critical point of F.

18. $\boxed{\text{d}}$ Since $f_x(20,10) = f_y(20,10) = 0$, the point $(20,10)$ is definitely a critical point. The discriminant of f at $(20,10)$ is

$$D(20,10) = f_{xx}(20,10)f_{yy}(20,10) - (f_{xy}(20,10))^2 = (-2)(-5) - 3^2 = 1 > 0.$$

The discriminant is positive, so $(20,10)$ is a relative minimum or maximum. Also, $f_{xx}(20,10) = -2 < 0$. So $(20,10)$ is a relative maximum by the Second Derivative Test.

19. $\boxed{\text{e}}$ We must have

$$1 = \int_0^1\int_0^1 f(x,y)\,dy\,dx = \int_0^1\int_0^1 Cxy\,dy\,dx = C\int_0^1 \frac{xy^2}{2}\Big|_{y=0}^{y=1} dx = C\int_0^1 \frac{x}{2}\,dx = C\frac{x^2}{4}\Big|_{x=0}^{x=1} = \frac{C}{4}.$$

Therefore we must have $C = 4$.

20. $\boxed{\text{d}}$ The solid is defined by the inequalities $-1 \le x \le 1$, $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$, $0 \le z \le 2$. The inequalities for x and y define the unit circular disk centered at the origin in the xy-plane. The inequality for z extends this disk to a circular cylinder of radius 1 and height 2 along the z-axis.