Part I Multiple Choice NO CALCULATOR ALLOWED

Name: Solutions Alfa: __________ Instructor:____________ Section:______

Instructions: No calculator is allowed for Part I of this exam. Fill in the top part of your Scantron sheet, including the bubbles for your alpha code. There is no extra penalty for wrong answers on Part I. There is room for your work on this exam. Fill in your answers on the bubble sheet. When you are done with Part I, hand in your bubble sheet and this exam to your instructor, who will give you Part II. You can use your calculator for Part II. But you cannot return to Part I.

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1. Find the length of the vector $\mathbf{a} - \mathbf{b}$ if $\mathbf{a} = \mathbf{i} + 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j}$.

(a) 0
(b) $\sqrt{2}$
(c) 2
(d) $2\sqrt{2}$
(e) 4

$a = \langle 1, 0, 2\rangle \quad b = \langle 1, -2, 0\rangle$

$a - b = \langle 0, 2, 2\rangle$

$|a-b| = \sqrt{0^2 + 2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

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2. The figure shows parallelogram $PQRS$ with base PQ and height QS, both of length 4. What is the magnitude of the cross product $\overrightarrow{PQ} \times \overrightarrow{PS}$?

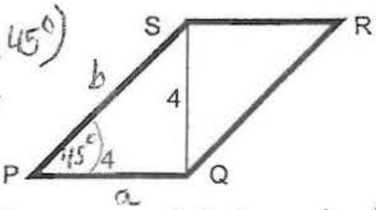
$\boxed{\text{Let } \overrightarrow{PQ} = a,\ \overrightarrow{PS} = b}$

3 ways to solve:

(a) $4\sqrt{2}$
(b) 8
(c) $8\sqrt{2}$
(d) 16
(e) $16\sqrt{2}$

I) $|a \times b|$ = area of parallelogram = base * height = $4 * 4 = 16$

II) $|a \times b| = |a|\,|b| \sin\theta = (4)(4\sqrt{2})(\sin 45^\circ) = 4 \cdot 4\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 16$



III) If P is point $(0,0,0)$, Q is point $(4,0,0)$ and S is the point $(4,0,4)$, then

$a = \langle 4-0, 0-0, 0-0\rangle = \langle 4,0,0\rangle$

$b = \langle 4-0, 0-0, 4-0\rangle = \langle 4,0,4\rangle$

$a \times b = \begin{vmatrix} i & j & k \\ 4 & 0 & 0 \\ 4 & 0 & 4 \end{vmatrix} = \langle 0, -16, 0\rangle$ and $|a\times b| = \sqrt{0^2 + (-16)^2 + 0^2} = 16$

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3. What surface is defined by the equation $r^2 + z^2 = 100$ in cylindrical coordinates?

$x^2 + y^2 + z^2 = 100$

sphere radius 10

(a) plane
(b) cylinder
(c) cone
(d) circular paraboloid
(e) sphere

4. The position of a particle in the plane is $\mathbf{r}(t) = \langle e^{2t}, e^t - 1\rangle$. Find the speed of the particle when it passes through the point $(1, 0)$.

(a) $\sqrt{5}$
(b) $2e^2$
(c) $2e^2 + 1$
(d) $\sqrt{e^4 + 1}$
(e) $\sqrt{4e^4 + 1}$

$v(t) = r'(t) = \langle 2e^{2t}, e^t \rangle$

Find t: $e^{2t} = 1$ and $e^t - 1 = 0$; $\therefore t = 0$ when the particle pass through point $(1,0)$

$v(0) = \langle 2e^{2\cdot 0}, e^0 \rangle = \langle 2, 1 \rangle$

speed @ $t=0$ $= |v(0)| = \sqrt{2^2 + 1^2} = \sqrt{5}$

5. Suppose that $\mathbf{r}'(t) = \langle 2t, 3t^2\rangle$ and $\mathbf{r}(1) = \langle 1, 4\rangle$. Find $\mathbf{r}(2)$.

(a) $\langle 0, 3\rangle$
(b) $\langle 2, 8\rangle$
(c) $\langle 2, 10\rangle$
(d) $\langle 4, 8\rangle$
(e) $\langle 4, 11\rangle$

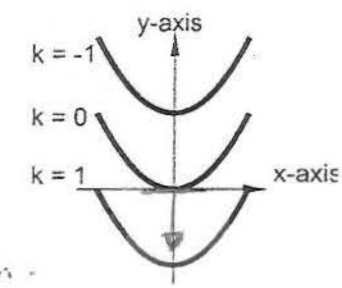
$r(t) = \langle \frac{2t^2}{2} + C_1, \frac{3t^3}{3} + C_2 \rangle$
$= \langle t^2 + C_1, t^3 + C_2 \rangle$

$1^2 + C_1 = 1$ $\quad 1^3 + C_2 = 4$
$C_1 = 0$ $\quad C_2 = 3$

$r(t) = \langle t^2 + 0, t^3 + 3 \rangle$
$r(2) = \langle 4, 11 \rangle$

6. The contour map shows the level curves $f(x, y) = k$ for $k = -1, 0, 1$. Which vector could be the gradient of f at the origin?

(a) **i** (b) −**i** (c) **j** (d) −**j** (e) **i**+2**j**



• in direction of increasing values of k.

• $\perp$ to tangent drawn @ origin.

7. Compute $F_{xy}(2, 3)$ if $F(x, y) = x^3y^2$.

(a) 6
(b) 18
(c) 36
(d) 48
(e) 72

$F_x = 3x^2y^2$

$F_{xy} = 3x^2(2y) = 6x^2y$

$F_{xy}(2,3) = 6(2)^2 \cdot 3 = 72$

8. Let $f(x, y) = e^x \sin(y) + (x-1)^2$. Let $\mathbf{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$. Find the directional derivative of f at the origin in the direction of $\mathbf{u}$.

(a) $-\frac{2}{5}$
(b) 0
(c) $\frac{2}{5}$
(d) 1
(e) $\frac{7}{5}$

$D_u f = \nabla f \cdot u$

$\nabla f = \langle e^x \sin y + 2(x-1), e^x \cos y \rangle$

$\nabla f(0,0) = \langle e^0 \sin 0 + 2(0-1), e^0 \cos 0 \rangle = \langle -2, 1 \rangle$

$D_u f = \langle -2, 1 \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle$

$= -2\left(\frac{3}{5}\right) + 1\left(\frac{4}{5}\right) = \frac{-6}{5} + \frac{4}{5} = \frac{-2}{5}$

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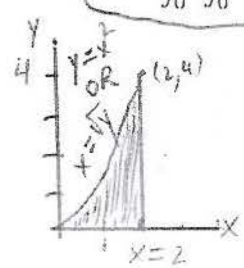
9. Reverse the order of integration to obtain a double integral that is equivalent to

$$\int_0^4 \int_{\sqrt{y}}^2 f(x, y)\, dx\, dy.$$

(a) $\int_{\sqrt{y}}^2 \int_0^4 f(x,y)\, dy\, dx$ (b) $\int_0^2 \int_{x^2}^4 f(x,y)\, dy\, dx$

(c) $\int_0^2 \int_0^{x^2} f(x,y)\, dy\, dx$ (d) $\int_0^2 \int_0^4 f(x,y)\, dy\, dx$ (e) $\int_0^4 \int_{x^2}^2 f(x,y)\, dy\, dx$



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10. Let R be the ring-shaped region between the two circles with polar equations $r = 1$ and $r = 2$. Evaluate the double integral

$$\iint_R 12(x^2 + y^2)\, dA.$$

$x^2 + y^2 = r^2$

(a) 3π
(b) 18π
(c) 36π
(d) 56π
(e) 90π

$\int_0^{2\pi} \int_1^2 r * 12 r^2\, dr\, d\theta =$

$\int_0^{2\pi} 1\, d\theta * \int_1^2 12 r^3\, dr$

$\theta \Big[_0^{2\pi}$ $\frac{12 r^4}{4} \Big[_1^2$

$(2\pi - 0)$ $3r^4 \Big[_1^2 =$

(2π) $*(48-3)$ $= 2\pi * 45 = 90\pi$

11. Find the cosine of the angle between the vectors $\langle 2, -2, 1\rangle$ and $\langle 2, 1, 2\rangle$.

a b

(a) 0
(b) $\frac{1}{9}$
(c) $\frac{1}{3}$
(d) $\frac{4}{9}$
(e) $\frac{4}{3}$

$a \cdot b = |a||b| \cos\theta$

$a \cdot b = 2(2) + (-2)(1) + 1(2) = 4$; $|a| = \sqrt{2^2 + (-2)^2 + 1^2} = 3$

$|b| = \sqrt{2^2 + (1)^2 + (2)^2} = 3$

$\frac{a \cdot b}{|a|\,|b|} = \cos\theta$

$\frac{4}{3*3} = \cos\theta$

12. Which one of the five given planes is parallel to the line with parametric equations $x = t, \quad y = 2t, \quad z = 3t$?

(a) $x + 2y + 3z = 14$
(b) $x + 2y - 3z = 0$
(c) $y - 2z = -1$
(d) $x + \frac{y}{2} + \frac{z}{3} = 0$
(e) $3x - 3y + z = 10$

$N = \langle 3, -3, 1\rangle$

line → parallel vector $\underset{V}{\langle 1, 2, 3\rangle}$

plane $N = \langle a, b, c\rangle$

dot product $V \cdot N = 0$

$\overset{V}{\langle 1, 2, 3\rangle} \cdot \overset{N}{\langle a, b, c\rangle} = 0$

only answer e satisfies the equation

$\overset{V}{\langle 1, 2, 3\rangle} \cdot \overset{N}{\langle 3, -3, 1\rangle} = 1(3) + 2(-3) + 3(1) = 0$

13. Which integral represents the length of the curve $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t), 3t\rangle$ for $0 \le t \le \pi$

(a) $\int_0^\pi 5\, dt$ (b) $\int_0^\pi 4\cos(t) + 4\sin(t) + 3t\, dt$ (c) $\int_0^\pi -4\sin(t) + 4\cos(t) + 3\, dt$

(d) $\int_0^\pi \sqrt{4\cos(t) + 4\sin(t) + 3t}\, dt$ (e) $\int_0^\pi (-4\sin(t))^2 + (4\cos(t))^2 + (3t)^2\, dt$

$L = \int_0^\pi |r'(t)|\, dt$

$= \int_0^\pi 5\, dt$

$r'(t) = \langle -4\sin t, 4\cos t, 3\rangle$

$|r'(t)| = \sqrt{(-4\sin t)^2 + (4\cos t)^2 + 3^2}$

$= \sqrt{16\sin^2 t + 16\cos^2 t + 9} = \sqrt{16+9} = \sqrt{25}$

$= 5$

14. Suppose $A = xy$, and $\frac{dx}{dt} = -3$, and $\frac{dy}{dt} = 4$. Find $\frac{dA}{dt}$ when $x = 5$ and $y = 6$.

(a) 2
(b) 9
(c) 18
(d) 38
(e) 39

$$\frac{dA}{dt} = \frac{dA}{dx}\frac{dx}{dt} + \frac{dA}{dy}\frac{dy}{dt}$$
$$= (y)(-3) + (x)(4)$$
when $x=5, y=6$ $$= (6)(-3) + (5)(4)$$
$$= 2$$

$$\frac{dA}{dx} = y \qquad \frac{dA}{dy} = x$$

15. Suppose $f(1,2) = 5$, $f_x(1,2) = 3$, and $f_y(1,2) = 4$. Estimate $f(1.1, 1.9)$ using the linear approximation (linearization) of f near $(1,2)$.

$x_0\ y_0\ z_0$

(a) 4.7
(b) 4.9
(c) 5.1
(d) 5.3
(e) 5.5

$$z - z_0 = f_x(x-x_0) + f_y(y-y_0)$$
$$z = z_0 + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$
$$z = 5. + 3(x-1) + 4(y-2)$$
$$f(1.1,1.9) \approx 5 + 3(1.1-1) + 4(1.9-2)$$
$$\approx 5 + .3 - .4 \approx 4.9$$

16. Find the tangent plane to the surface $x^2 + y^2 + xe^z = 6$ at the point $(1,2,0)$.

(a) $3x + 4y + z = 11$
(b) $3x + 2y + z = 5$
(c) $2x + 4y + z = 10$
(d) $2x + 2y + 3z = 6$
(e) $2x - 2y + 2z = 0$

$\nabla f(1,2,0)$ = Normal vector
$$\nabla f = \langle 2x + e^z, 2y, xe^z \rangle$$
$$\nabla f(1,2,0) = \langle 2(1)+e^0, 2(2), 1e^0 \rangle = \langle 3,4,1 \rangle$$
tangent plane @ pt $(1,2,0)$
$$3(x-1) + 4(y-2) + 1(z-0) = 0$$
$$3x + 4y + z = 11$$

17. How many critical points does the function $F(x, y) = x^2 + y^2 - 6y + 10$ have?

(a) 0
(b) 1
(c) 2
(d) 3
(e) more than 3

$F_x = 2x = 0$ $\quad$ $F_y = 2y - 6 = 0$

$x = 0$ $\quad$ $y = 3$

one critical point

$F(0,3) = 1$

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18. Let $f(x, y)$ be a function of two variables with $f_x(20, 10) = f_y(20, 10) = 0$. Suppose $f_{xx}(20, 10) = -2$, $f_{yy}(20, 10) = -5$, and $f_{xy}(20, 10) = 3$. How should we classify the point $(20, 10)$?

(a) not a critical point
(b) a saddle point
(c) a relative minimum
(d) a relative maximum
(e) none of the above

$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} -2 & 3 \\ 3 & -5 \end{vmatrix} = (-2)(-5) - 3^2 = 1$

$D > 0$ and $f_{xx} < 0$

relative max

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19. Suppose that $f(x, y) = Cxy$ is a joint density function for the random variables X and Y with $0 \leq x \leq 1$ and $0 \leq y \leq 1$. What is the value of the constant C?

(a) $\frac{1}{4}$
(b) $\frac{1}{2}$
(c) 1
(d) 2
(e) 4

$1 = \int_0^1 \int_0^1 Cxy \, dy \, dx$

$1 = C * \int_0^1 x \, dx * \int_0^1 y \, dy$

$\frac{x^2}{2}\Big[_0^1 \quad \frac{y^2}{2}\Big[_0^1$

$1 = C * \left(\frac{1}{2}\right) * \left(\frac{1}{2}\right)$

$1 = C\left(\frac{1}{4}\right)$

$4 = C$

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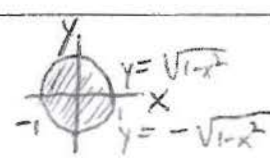
20. Which solid has volume described by the triple integral

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^2 1 \, dz \, dy \, dx \, ?$$



(a) sphere
(b) hemisphere
(c) cone
(d) cylinder
(e) cube

$Z = 2 \rightarrow$

$z = 0 \rightarrow$

End of Part I

When you are done with Part I, hand in your bubble sheet and the exam itself to your instructor who will give you Part II. You can use your calculator for Part II, but cannot return to Part I.

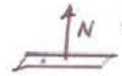
Part II solutions

1) (a)

vector $a = \langle 1-1, 0-2, -1-3 \rangle = \langle 0,-2,-4 \rangle$

vector $b = \langle 2-1, 1-0, 1--1 \rangle = \langle 1,1,2 \rangle$

Normal vector $N = a \times b = \begin{vmatrix} i & j & k \\ 0 & -2 & -4 \\ 1 & 1 & 2 \end{vmatrix} = \langle 0,-4,2 \rangle$

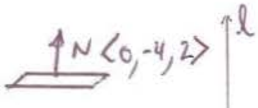


using pt $(1,2,3)$, equation of plane: $0(x-1) - 4(y-2) + 2(z-3) = 0$

or

$\boxed{2y - z = 1}$

(b)



$N = \langle 0,-4,2 \rangle$ is <u>parallel</u> to line l.

parametric equations of line l: $x = 1 + 0t,\ y = 1 - 4t,\ z = 1 + 2t$

vector equation of line l: $\boxed{r(t) = \langle 1, 1-4t, 1+2t \rangle}$

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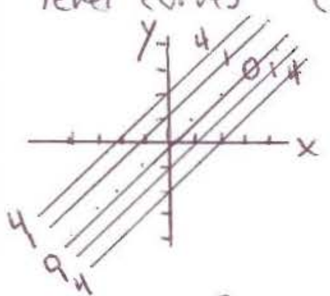
2) level curves $(x-y)^2 = k$:

(a)



$(x-y)^2 = 0$

$x - y = \pm 0$

$\underline{y = x}$

for $k = 0$

$(x-y)^2 = 1$

$x - y = \pm\sqrt{1}$

$x - y = 1$ or $\underline{y = x - 1}$

$x - y = -1$ or $\underline{y = x + 1}$

for $k = 1$

$(x-y)^2 = 4$

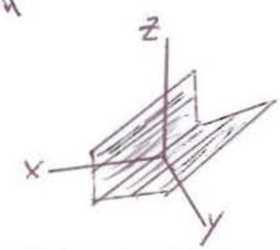
$x - y = \pm\sqrt{4}$

$x - y = 2$ or $\underline{y = x - 2}$

$x - y = -2$ or $\underline{y = x + 2}$

for $k = 4$

(b)



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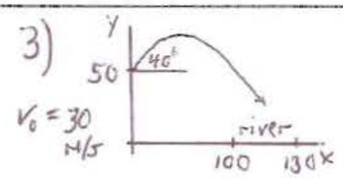
3)



$x = V_0(\cos\alpha)t \qquad y = V_0(\sin\alpha)t - \frac{1}{2}gt^2 + 50$

$0 = 30(\sin 40°)t - 4.9t^2 + 50$

$t = 5.7195157$ seconds (via calculator)

$x = 30(\cos 40°) * (5.7195157) = 131.44$ meters. The ball clears the river by 1.44 m.

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4) (a) A 22-year old who swims 10,000 meters daily can swim 400 meters in 230 seconds.

(b) A 22-year old swimmer's 400-meter time decreases .001 seconds per an increase of 1 meter swum daily (or 1 second per 1000 m swum daily).

(c) A swimmer's 400-meter time increases 2.1 seconds per an increase of 1-year in age (for a swimmer who swims 10,000 meters daily).

5) a) $\frac{df}{dt} = \frac{df}{dT}\frac{dT}{dt} + \frac{df}{d\rho}\frac{d\rho}{dt}$

$= (4)(3) + (.17)(-2) = \boxed{11.66\ M/s/s}$

b) $z = z_0 + f_T(T-T_0) + f_p(p-p_0)$

$f(T,p) = f(7,5) + \frac{df}{dt}(T-7) + \frac{df}{d\rho}(p-5)$

$f(7.3,4.5) \approx 1500 + 4(7.3-7) + .17(4.5-5) \approx \boxed{1501.115\ M/s}$

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6) $g_x = x^2 - 1 = 0$ $\quad x = \pm 1$ $\qquad g_y = y^2 - 4 = 0$ $\quad y = \pm 2$

$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2x & 0 \\ 0 & 2y \end{vmatrix} = \boxed{4xy}$

CPs	D	
$g(1,2) = -5$	>0	$f_{xx} > 0$ local min.
$g(1,-2) = 5.\bar{6}$	<0	s.p.
$g(-1,2) = -3.\bar{6}$	<0	s.p.
$g(-1,-2) = 7$	>0	$f_{xx} < 0$ local max.

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7) Enjoyment $= f(x,y)$ $\qquad$ constraint g

$f(x,y) = xy$ $\qquad g(x,y) = 30x + 20y = 100$

$\nabla f = \langle y, x\rangle \qquad \lambda \nabla g = \langle 30\lambda, 20\lambda\rangle$

$y = 30\lambda$; $x = 20\lambda$; $30x + 20y = 100$

$30(20\lambda) + 20(30\lambda) = 100$

$1200\lambda = 100 \qquad \lambda = 1/12$

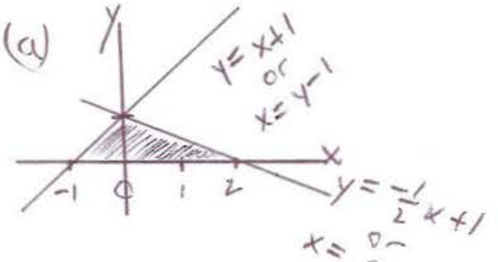
$y = 30(1/12) = 2.5 \quad x = 20(1/12) = 5/3$

The most "enjoyment" you can buy for \$100

(Using 2.5 liters of coke and 5/3 lb. of candy)

$= f(5/3, 2.5) = 5/3 * 2.5 = \boxed{25/6}$.

8) (a)



$$\iint_R x\, dA =$$

$$= \boxed{\int_{-1}^{0}\int_{0}^{x+1} x \; dy\, dx + \int_{0}^{2}\int_{0}^{-\frac{1}{2}x+1} x \; dy\, dx}$$

OR

$$= \boxed{\int_0^1 \int_{y-1}^{2-2y} x \; dx\, dy}$$

$$= \frac{x^2}{2}\Big[_{y-1}^{2-2y}$$

$$\frac{1}{2}\left[(2-2y)^2 - (y-1)^2\right]$$

$$\frac{1}{2}\left[4-8y+4y^2 - y^2 + 2y - 1\right]$$

$$\frac{1}{2}\int_0^1 3y^2 - 6y + 3 \; dy$$

$$\frac{1}{2}\left[y^3 - 3y^2 + 3y\right]\Big[_0^1$$

$$\frac{1}{2}\left[(1-3+3) - (0)\right] = \boxed{\frac{1}{2}}$$

(b)

$$\int_0^{2\pi}\int_0^1 r * e^{r^2} \, dr\, d\theta =$$

$$\int_0^{2\pi} 1 \, d\theta \quad \Big| \quad \int_0^1 r e^{r^2} dr$$

u-sub $\boxed{u} = r^2$

$du = 2r\,dr$

$\boxed{\frac{1}{2}du} = r\,dr$

$$= \theta\Big[_0^{2\pi} \quad \Big| \quad = \int \frac{1}{2} e^u \, du$$

$$= 2\pi \quad * \quad \Big| \quad = \frac{1}{2}e^u\Big[$$

$$= \frac{1}{2}e^{r^2}\Big[_0^1$$

$$= \left(\frac{1}{2}e^1 - \frac{1}{2}e^0\right)$$

$$2\pi \; * \; \frac{1}{2}(e-1) \quad = \quad \boxed{\pi(e-1)}$$