SM316
SOLUTIONS to
SPRING 2014 FINAL

1. X = # of H's in 4 losses

(a) $|S| = \underline{2} \cdot \underline{2} \cdot \underline{2} \cdot \underline{2} = 2^4 = 16$

(b) X is a binomial r.v w. a success meaning you get H

(i) Coin Fair $\Rightarrow P(H) = p = .5$, $n = 4$

$P(1 \le X < 4) = .875$

(ii) $P(H) = .2 = p$

$P(1 \le X < 4) = .5888$

Both found by CALC.

(c) $P(H) = p$, and PMF $= f(x)$

$= P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$ (fmla binomial) $\Rightarrow$ $(n = 4, p = p)$

$P(2 \le X < 4) = f(2) + f(3)$

$= \binom{4}{2} p^2 (1-p)^2 + \binom{4}{3} p^3 (1-p)$

$= 6p^2(1-p^2) + 4p^3(1-p)$

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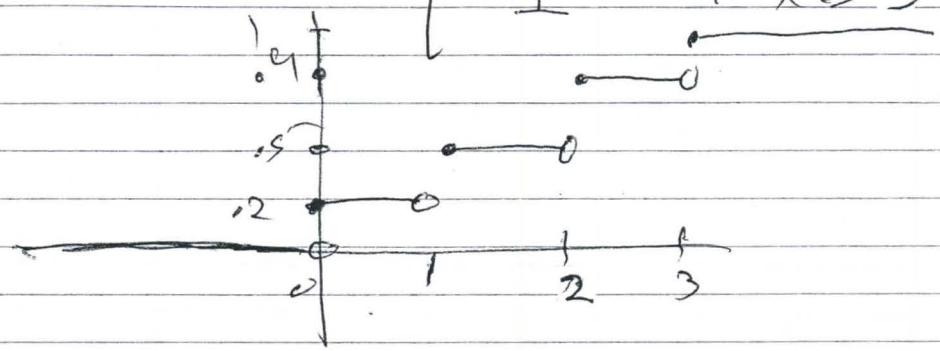
2. X = # of red lights

(a) For any discrete r.v. X w pmf $f(x)$, $\sum_{\text{all } x} f(x) = 1 \Rightarrow$

$.2 + .3 + .4 + k = 1 \Rightarrow k = .1$

(b) Since X takes on 4 values with nonzero prob, it takes 5 steps to find CDF

$$F(x) = P(X \le x) = \begin{cases} 0 & \text{IF } x < 0 \\ .2 & \text{IF } 0 \le x < 1 \\ .5 & \text{IF } 1 \le x < 2 \\ .9 & \text{IF } 2 \le x < 3 \\ 1 & \text{IF } x \ge 3 \end{cases}$$



(c) $E(X) = \sum_{\text{all } x} x f(x) = 0(.2) + 1(.3) + 2(.4) + 3(.1)$

$= 1.4$

$Var(X) = E(X^2) - (E(X))^2$

$E(X^2) = \sum_{\text{all } x} x^2 f(x) = 0^2(.2) + 1^2(.3) + 2^2(.4) + 3^2(.1)$

$= 2.8$

So $Var(X) = 2.8 - (1.4)^2 = 2.8 - 1.96$

$= .84$

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3. $\bar{X} = \sum_{i=1}^{30} X_i$

(a) $\bar{X}$ is normal (approx) by CLT

(b) (Recall $E(\bar{X}) = \mu$, $StDev(\bar{X}) = \frac{\sigma}{\sqrt{n}}$)

So $E(\bar{X}) = 1.4$

$$StDev(\bar{X}) = \frac{\sqrt{.84}}{\sqrt{30}} = .1673$$

(c) So $\bar{X}$ is $N(x; 1.4, .1673) \Rightarrow$

$$P(\bar{X} > 2) = .00017 \approx .0002$$

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4. X = time he waits for train (in min.'s)

PDF $$f(x) = \begin{cases} 0 & \text{IF } x < 0 \\ \frac{1}{10} e^{-x/10} & \text{IF } x \geq 0 \end{cases}$$

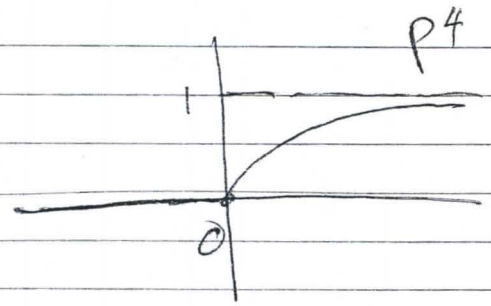
(a) Since $f(x)$ partitions $(-\infty, \infty)$ into 2 subintervals, it takes 2 steps:

$$F(x) = P(X \leq x) = \begin{cases} 0 & \text{IF } x < 0 \\ \int_{-\infty}^{x} f(t)\, dt & \text{IF } x \geq 0 \end{cases}$$

But $$\int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{0} 0\, dt + \int_{0}^{x} \frac{1}{10} e^{-t/10}\, dt$$

$$= -e^{-t/10}\Big|_{t=0}^{x} = 1 - e^{-x/10}$$

So $$F(x) = \begin{cases} 0 & \text{IF } x < 0 \\ 1 - e^{-x/10} & \text{IF } x \geq 0 \end{cases}$$



(b) (i) $P(X > 3) = 1 - P(X \le 3)$

$$= 1 - (1 - e^{-3/10}) = e^{-.3}$$

$$= .7408$$

(Can also be done with an integral $\int_3^\infty \frac{1}{10} e^{-x/10}\,dx$)

(ii) $P(X > 6 \mid X > 3) = \dfrac{P((X > 6) \cap (X > 3))}{P(X > 3)}$

$$= \frac{P(X > 6)}{P(X > 3)} = \frac{1 - (1 - e^{-6/10})}{1 - (1 - e^{-3/10})} = \frac{e^{-.6}}{e^{-.3}} = e^{-.3}$$

$$= .7408$$

(iii) What this says is the prob. that he has to wait at least 3 more min's given that he has already waited 3 min's is <u>equal</u> to the probability that he has to wait at least 3 min's. We say the r.v. has forgotten that he has already waited 3 min's (This is an important property of the r.v. called the memoryless property!)

4(c) If you observe that this is an exp r.v. with $\beta = 10$, then $E(X) = 10$ and $Var(X) = 100$. Otherwise

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{0}^{\infty} \frac{x}{10} e^{-x/10}\,dx = 10$$

$$Var(X) = \int_{-\infty}^{\infty} (x-10)^2 f(x)\,dx$$

$$= \frac{1}{10}\int_{0}^{\infty} (x-10)^2 \cdot e^{-x/10}\,dx = 100$$

(Where the calc. has been used to do the integration.)

5. (a) $P(M_1) = 2P(M_2)$. But all bottles are filled by either M_1 or M_2 $\Rightarrow$ $P(M_1) + P(M_2) = 1$

$\Rightarrow P(M_2) = \frac{1}{3}$ and $P(M_1) = \frac{2}{3}$

Since M_1 and M_2 are normal r.v's w.

$M_1 \quad N(x; 16.21, .14)$ $\Longrightarrow$

$M_2 \quad N(x; 16.12, .07)$

$P(fill < 15.96 \mid M_1) = P(M_1 < 15.96) = .03707$
$\approx .0371$

$P(fill < 15.96 \mid M_2) = P(M_2 < 15.96) = .01114$
$\approx .0111$

(Here we are just using M_1, M_2 as though they are normal r.v's & using calc. to get prob's.)

p6

	M_1	M_2	
fill < 15.96	.0247	.0037	.0284
fill ≥ 15.96			
	2/3	1/3	1

$P((\text{fill} < 15.96) \cap M_1) =$
$P(\text{fill} < 15.96 \mid M_1) \cdot P(M_1)$
$= (.0371)(\frac{2}{3})$
$= .02473$
$\approx .0247$

$P((\text{fill} < 15.96) \cap M_2) =$
$(.0111)(\frac{1}{3}) = .0037$

$\therefore P(\text{fill} < 15.96) = .0284$

(b) $P(M_1 \mid \text{fill} < 15.96) = \dfrac{P(M_1 \cap \text{fill} < 15.96)}{P(\text{fill} < 15.96)}$

$= \dfrac{.0247}{.0284} = .8697$

(You can fill in the other cells of the TVD by subtraction, but it is unnec. for this problem.)

6. $n_A = 60$, $\mu_A = 1827$, $\sigma_A = 168$

$n_B = 180$, $\mu_B = 1658$, $\sigma_B = 225$

(a) $\bar{X}_A$ & $\bar{X}_B$ are normal by CLT

$\bar{X}_A - \bar{X}_B$ is normal by Reproductive Property (of the normal)

P7

(b) $E(\bar{X}_A - \bar{X}_B) = E(\bar{X}_A) - E(\bar{X}_B)$

$= 1827 - 1658 = 169$

$Var(\bar{X}_A - \bar{X}_B) = Var(\bar{X}_A) + Var(\bar{X}_B)$

$= \frac{(168)^2}{60} + \frac{(225)^2}{180}$

$\Rightarrow StDev(\bar{X}_A - \bar{X}_B) = \sqrt{\frac{(168)^2}{60} + \frac{(225)^2}{180}}$

$= 27.416236$

≈ 27.4162

Thus $\bar{X}_A - \bar{X}_B \sim N(x; 169, 27.4162)$

$P(\bar{X}_A - \bar{X}_B \leq 100) = .00592199$

$\approx .0059$

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7. (a) $\sigma^2 = 48$, $n = 41$ $S^2 = 80$

$P(S^2 \geq 80)$

$\chi^2 = \frac{(n-1)S^2}{\sigma^2}$ is a chi-squared r.v. w $df = 40$

$\Rightarrow P(S^2 \geq 80) = P\left(\chi^2 \geq \frac{(40)(80)}{48}\right)$

$= P(\chi^2 \geq 66.6667)$

$= .0051$

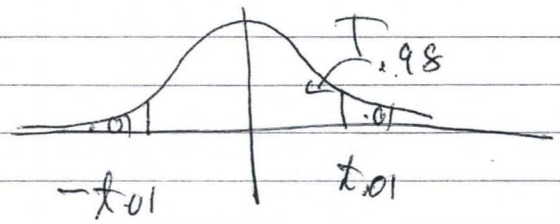
Since this probability is so small, WE REJECT claim of ETS.

7(b) $n=7$ $\bar{x} = 62.88$

$s = 5.4838$

2-sided 98% CI $\Rightarrow 1-\alpha = .98 \Rightarrow \alpha = .02 \Rightarrow$

$\frac{\alpha}{2} = .01$ Since pop. var is unknown, we must use t-dist. $\Rightarrow df = 6$



$P(T \le t_{.01}) = .99$

By Calc (INV-t),

$\Rightarrow t_{.01} = 3.1426684$

$\cong 3.1427$

$\therefore$ 98% CI is $62.88 \pm \frac{(3.1427)(5.4838)}{\sqrt{7}}$

62.88 ± 6.5138

$\Rightarrow$ CI is $56.3662 \le \mu \le 69.3938$

MATRIX THEORY

8 (a) Suppose $c_1\begin{pmatrix}1\\2\\5\end{pmatrix} + c_2\begin{pmatrix}2\\5\\1\end{pmatrix} + c_3\begin{pmatrix}1\\5\\2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$\Rightarrow$
$$\begin{aligned} c_1 + 2c_2 + c_3 &= 0 \\ 2c_1 + 5c_2 + 5c_3 &= 0 \\ 5c_1 + c_2 + 2c_3 &= 0 \end{aligned}$$

To solve this system we row reduced its augmented matrix

$$\text{rref}\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 5 & 5 & 0 \\ 5 & 1 & 2 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \Rightarrow c_1 = c_2 = c_3 = 0$$

$\therefore$ vec's are l.i.

(b) The set $\{X_1, X_2, \ldots, X_k\}$ is l.d if there exist scalars $c_1, c_2, \ldots, c_k$ not all zero s.t. $c_1 X_1 + c_2 X_2 + \cdots + c_k X_k = 0$. This is a homogeneous system of n-equations (since X_i are $(n \times 1)$ column's) in k unknowns (the c_i). What this says in terms of solving this system is that this system has a nontrivial (nonzero) solution.

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9. (a) $$\text{rref}(A) = \begin{pmatrix} 1 & 2 & 0 & -5/2 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ rank of A is 2

(b) This says these vectors are l.d

(c) If one considers $AX = 0$, then the system well have 2 free variables $\therefore \infty$ many soln's. So it well have a nonzero soln

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10. $$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 2 & 3 & -1 & 3 \\ 5 & 7 & 1 & 7 \end{array}\right) \xrightarrow[-5R_1+R_3]{-2R_1+R_2} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & -7 & 1 \\ 0 & 2 & -14 & 2 \end{array}\right) \xrightarrow[-1R_1+R_1]{-2R_1+R_3}$$

#10 cont'd $\left(\begin{array}{ccc|c} 1 & 0 & 10 & 0 \\ 0 & 1 & -7 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$ rref

$$\begin{aligned} x + 10z &= 0 \\ y - 7z &= 1 \\ z &= z \text{ (free)} \end{aligned} \Rightarrow \begin{aligned} x &= -10z \\ y &= 1 + 7z \\ z &= z \text{ (free)} \end{aligned}$$

(b) It has ∞ many soln's.

$z = 0 \Rightarrow x = 0, y = 1$; $z = 1 \Rightarrow x = -10, y = 8$

(c) This system can not be solved by Cramer's Rule b/c the det of the coeff matrix is 0 - we knew this b/c the row reduction to rref gave a row of 0's. So there is no inverse, so det must be 0.

11. (a) The eval's are the soln's of char poly:

$$\det(C - \lambda I) = \begin{vmatrix} 2-\lambda & 2 & -5 \\ 3 & 7-\lambda & -15 \\ 1 & 2 & -4-\lambda \end{vmatrix} = -(\lambda-1)(\lambda^2 - 4\lambda + 3)$$

(bx CALC.)

$$= -(\lambda-1)^2(\lambda-3)$$

So eval's are $\lambda_1 = 1, 1$ (AM=2), $\lambda_2 = 3$ (AM=1)

$\lambda_1 = 1$, $\bar{X}_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is a soln to $(A - \lambda_1 I)\bar{X}_1 = 0 \Rightarrow$

$$\text{rref}(A - \lambda_1 I | 0) = \left(\begin{array}{ccc|c} 1 & 2 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$$

$x_1 + 2x_2 - 5x_3 = 0$ $\qquad$ $x_1 = -2x_2 + 5x_3$

$x_2 = x_2$ (free) $\Rightarrow$ $x_2 = x_2$ (free)

$x_3 = x_3$ (free) $\qquad$ $x_3 = x_3$ (free)

Since we have 2 free var's, we can find 2 l.i. e-vec's corresponding to $\lambda_1 = 1$ (i.e. GM = 2) as follows: First take $x_2 = 1$, $x_3 = 0$ $\Rightarrow$ $x_1 = -2$, $X_{1,1} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

Take $x_2 = 0$, $x_3 = 1$ $\Rightarrow$ $x_1 = 5$, $X_{1,2} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$

(These vec's are clearly l.i.)

For $\lambda_2 = 3$, X_2 is a soln to $(A - \lambda_2 I) X_2 = 0 \Rightarrow$

$$\text{rref}(A - \lambda_2 I \mid 0) = \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$$

$x_1 = x_3$

$x_2 = 3x_3$ $\qquad$ $\therefore x_3 = 1$, $X_2 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$

$x_3 = x_3$ (free)

(b) Since we found 3 l.i. evec's, the matrix C is diagonalizable

(c) $P = \begin{pmatrix} -2 & 5 & 1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

s.t. $P^{-1} C P = D$

12(a) A is diagonalizable b/c it has distinct e-val's (the 3 different e val's give 3 l.i. e vec's)

(b) $B^T = \begin{pmatrix} 2 & -3 \\ -3 & 4 \end{pmatrix} = B$ $\therefore$ B is symmetric

$\Rightarrow$ B is diagonalizable

(c) $\det(B) = 8 - 9 = -1$, $\det(C) = 3 - 2 = 1$

Since $\det(B) \neq \det(C)$,

B and C can <u>NOT</u> be similar.