SM316 FINAL SOLUTIONS
SPRING 2012

1. $P(H) = p,\ P(T) = q = 1-p$

$n = 10$, Put $X = \#$ of H's

(a) $P(X=10) = p^{10}$

(b) $P(X \neq 10) = 1 - p^{10}$

$P(1\text{st H on }10\text{th toss}) = q^9 p$

(c) $P(1\text{st} \,\&\, \text{Last are H's} \,\&\, \text{in remaining } 8,\ X=3)$

$= p^2 \cdot \binom{8}{3} p^3 q^5 = 56 p^5 q^5$

(d) $P(= \#\text{ of H's} \,\&\, \text{T's}) \Rightarrow P(X=5) =$

(e) $\binom{10}{5} p^5 q^5 = 252 p^5 q^5$

(f) $p = \frac{1}{10}$; $E(X) = np = 10 \cdot \frac{1}{10} = 1$ H

$Var(X) = npq = 10 \cdot \frac{1}{10} \cdot \frac{9}{10} = \frac{9}{10}$

(g) $n = 100$

$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

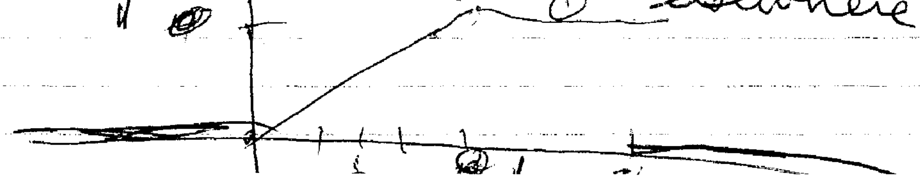
$= q^{100} + \binom{100}{1} p q^{99} + \binom{100}{2} p^2 q^{98}$

4950

$= q^{100} + 100 p q^{99} + \quad p^2 q^{98}$

(h) $p = 1/10 \Rightarrow P(X \leq 2) = .0019$

#2 (a) $f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 1 & \text{if } 1 \leq x \leq \frac{3}{2} \\ 0 & \text{elsewhere} \end{cases}$



CDF

(b) $F(x) = P(X \le x)$

Since this breaks $-\infty < x < \infty$ up into 4 subintervals $x<0$, $0 \le x \le 1$, $1 \le x \le \frac{3}{2}$, $y > \frac{3}{2}$

$\therefore$ It takes 4 steps to find the CDF

Step ① Let $x<0$

$$\Rightarrow F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0$$

Step ② Let $0 \le x \le 1$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{0} 0\,dt + \int_{0}^{x} t\,dt$$

$$= \frac{1}{2}t^2 \Big|_{t=0}^{x} = \frac{1}{2}x^2$$

Step ③ Let $1 \le x \le \frac{3}{2}$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{0} 0\,dt + \int_{0}^{1} t\,dt +$$

$$\int_{1}^{x} 1\,dt = \frac{1}{2} + (x-1)$$

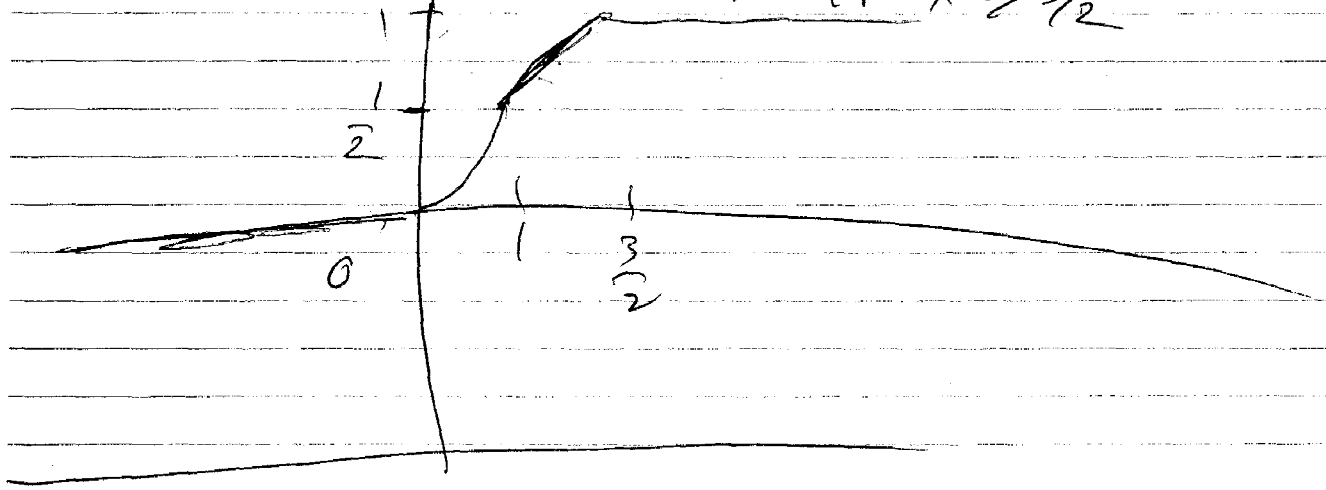
Step ④ Let $x \ge \frac{3}{2}$: $F(x) = \int^{x} f(t)\,dt =$

p3

$$F(x) = \underbrace{\int_{-\infty}^{0} 0\,dt}_{0} + \int_{0}^{1} t\,dt + \int_{1}^{3/2} 1\,dt + \underbrace{\int_{3/2}^{x} 0\,dt}_{0}$$

$$= \frac{1}{2} + \frac{1}{2} = 1 \Rightarrow$$

$$F(x) = \begin{cases} 0 & \text{IF } x < 0 \\ \frac{1}{2}x^2 & \text{IF } 0 \le x \le 1 \\ \frac{1}{2} + (x-1) = x - \frac{1}{2} & \text{IF } 1 \le x \le \frac{3}{2} \\ 1 & \text{IF } x \geq 3/2 \end{cases}$$



(c) $P\left(\frac{1}{2} \le X \le \frac{5}{4}\right) = F\left(\frac{5}{4}\right) - F\left(\frac{1}{2}\right)$

$$= \left(\frac{5}{4} - \frac{1}{2}\right) - \frac{1}{2}\cdot\left(\frac{1}{4}\right)$$

$$= \frac{3}{4} - \frac{1}{8} = \frac{5}{8} = .625$$

(d) $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{0}^{1} x\cdot x\,dx + \int_{1}^{3/2} x\cdot 1\,dx$

$$= \frac{x^3}{3}\Big|_0^1 + \frac{x^2}{2}\Big|_1^{3/2}$$

$$= \frac{1}{3} + \frac{5}{8} = \frac{23}{24} \approx .9583$$

(e) $Var(X) = \sigma^2 = E\left(\left(X - \cancel{a}\ \frac{23}{24}\right)\right)^2$

$$= \int_0^1 \left(x - \frac{23}{24}\right)^2 \cdot x\,dx + \int_1^{3/2} \left(x - \frac{23}{24}\right)^2 \cdot 1\,dx$$

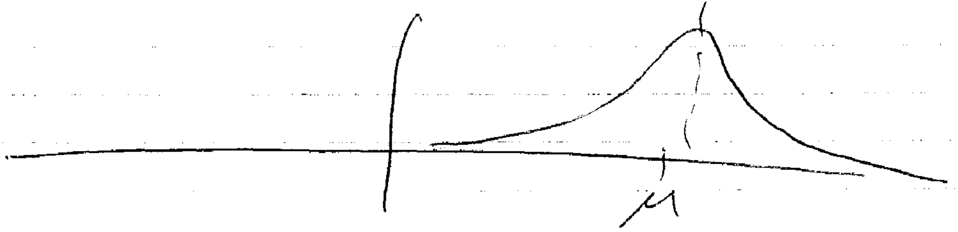
$$= \frac{71}{576} \approx .123264 \approx .1233$$

3. X normal $E(X) = \mu$, Std Dev$(X) = \sigma$

$n(x;\mu,\sigma)$

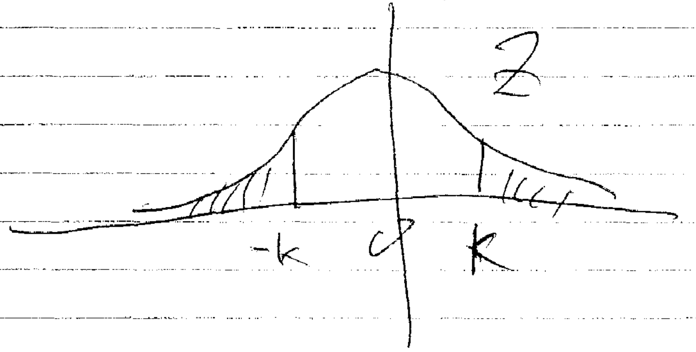
$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right), \quad -\infty < x < \infty$$

(a)



(b) $\mu = 0, \sigma = 1$; $n(z;0,1)$

$P(-k \le Z \le k)$
$= 2P(Z \le k) - 1$



By Symmetry
$P(Z \le -k) =$
$P(Z \ge k)$

Consider that the total area under this curve is 1, then

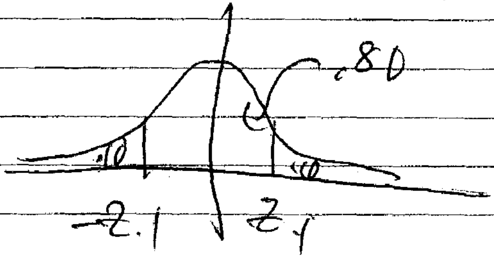
$$2P(Z \le k) - 1 = 2\big(P(Z \le -k) + P(-k \le Z \le k)\big)$$
$$- \big(P(Z \le -k) + P(-k \le Z \le k) + P(Z \ge -k)\big)$$

$$= 2P(Z \le -k) + 2P(-k \le Z \le k)$$
$$- \big(2P(Z \le -k) + P(-k \le Z \le k)\big)$$
$$= P(-k \le Z \le k).$$

(c) the CLT says that the sum of indep r.v. tends to being normal & the more you sum the closer to normal it becomes.

(d) 80% CI μ $\sigma^2 = 4$ $n = 25$
$\bar{X} = 24$

P6



$1-\alpha = .8$

$\alpha = .2$

$\frac{\alpha}{2} = .1$ $\sigma = 2$

$z_{.1} =$ 1.28155

(Area = .9)

$$\Rightarrow 2.4 \pm \frac{(1.28155)\cdot 2}{\sqrt{25}}$$

$2.4 \pm .51262$

$1.88738 \le \mu \le 2.91262$

or $1.8874 \le \mu \le 2.9126$

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4. $X = X_1 + X_2 + \cdots + X_{52}$

$E(X_i) = 200$

$StDev(X_i) = 20$

(a) X is normal by CLT

(b) $E(X) = (200)(52) = 10,400$

$Var(X) = (20)^2(52) = 20,800$

$\Rightarrow StDev(X) = \sqrt{20,800} \approx 144.22$

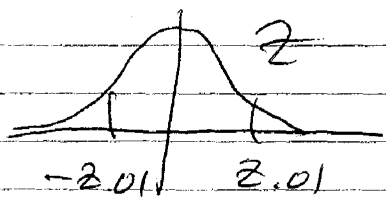
(c) $P(X > 11,000) = .000016$ $X = n(x; 10,400, 144.22)$

(d) $P(\bar{X} \geq k) = .9 \Rightarrow P(\bar{X} \leq k) = .1$

$k = 10,215.0$

#5. $\sigma = .45$ min, $n = 49$, $\bar{X} = 95.23$ min's

(a)



b/c $\sigma = .45$ known

$1-\alpha = .98 \Rightarrow \alpha = .02$

$\frac{\alpha}{2} = .01$

$\Rightarrow z_{.01} = 2.32635$

So 98% CI is

$$95.23 \pm \frac{(2.32635)(.45)}{\sqrt{49}}$$

$95.23 \pm .149551 \Rightarrow$

$95.0804 \leq \mu \leq 95.3796$ is the 98% CI

(b) ERROR $= |\bar{X} - \mu| \leq .149551$ 98% of the time

$\leq .1496$

(c) If he wants err $\leq .12 \Rightarrow$

$$n = \left(\frac{(2.32635)(.45)}{.12}\right)^2$$

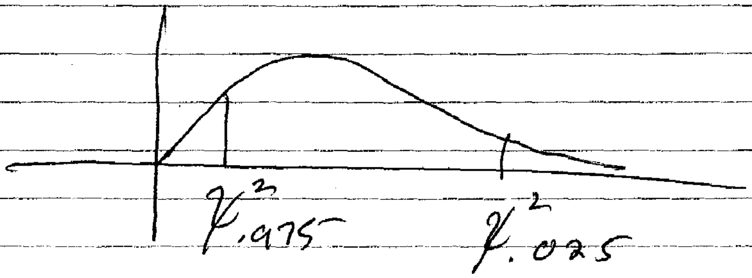
$= 76.1049$

$\Rightarrow n = 77$ – he has to sample 77 p 8

6 (a) $\mu = 5000m$

$\sigma^2 = 500$

$n = 16, \quad S^2 = 350$



$df = 15, \quad \chi^2_{.025} = 27.4884$

$\chi^2_{.975} = 6.26214$

$\chi^2 \text{ score} = \frac{(n-1)S^2}{\sigma^2} = \frac{(15)(350)}{500} = 10.5$

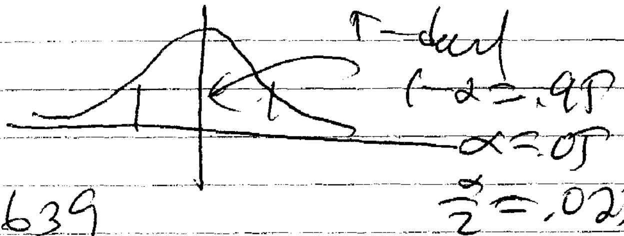
Since this falls between the 2 bounds
We conclude the variance is still $\sigma^2 = 500$

Thus this refutes the company claim.

(b) $n = 25 \quad \overline{X} = 660.25$ Since σ unknown

$S = 6.25$

we must use T-dist.



$\# df = 24 \quad \therefore t_{.025} = 2.0639$

$660.25 \pm \frac{(2.0639)(6.25)}{5}$

$660.025 \pm 2.57988 \Rightarrow$

∴ 95% 2-sided CI for MATH SAT

mean is $657.\text{[illegible]} \le \mu \le$ ~~[illegible]~~ (67012, 662.82985 written above)

~~[illegible]~~

$657.6701 \le \mu \le 662.8299$

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7 (a) $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 2 & -2 & 8 \\ 1 & 0 & 1 & 4 \end{array}\right)$

$\underbrace{\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{pmatrix}}_{A} \underline{X} = \begin{pmatrix} 0 \\ 8 \\ 4 \end{pmatrix}$ where $\underline{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

(b) $\det(A) = 1$

(c) System has a unique soln

(d) System has a unique soln.

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 2 & -2 & 8 \\ 1 & 0 & 1 & 4 \end{array}\right) \xrightarrow[-R_1+R_3]{-R_1+R_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 8 \\ 0 & -1 & 2 & 4 \end{array}\right)$$

$$\xrightarrow[R_2+R_3]{-R_2+R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -8 \\ 0 & 1 & -1 & 8 \\ 0 & 0 & 1 & 12 \end{array}\right) \xrightarrow{1 \cdot R_3+R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -8 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 12 \end{array}\right)$$

$\Longrightarrow x = -8, y = 20, z = 12$

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The Same EROs would find A^{-1}

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 2 & -2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[-R_1+R_3]{-R_1+R_2} \left(\begin{array}{ccc|ccc} 1 & +1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_2+R_3]{-R_2+R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right) \longrightarrow$$

$$\xrightarrow{R_3+R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -3 & 2 & 1 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right) \Rightarrow$$

$$A^{-1} = \begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & 1 \\ -2 & 1 & 1 \end{pmatrix}$$

8.(a) $$\text{rref}\begin{pmatrix} 1 & 2 & -2 \\ -1 & 0 & 2 \\ -1 & 4 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

l.d. <u>Not</u> l.i.

(b) $\begin{vmatrix} 9 & k \\ k & 1 \end{vmatrix} = 9-k^2$ ∴ it will have a unique soln if $k \neq \pm 3$

$\Rightarrow x = \begin{vmatrix} 1 & k \\ 0 & 1 \end{vmatrix} \Big/ (9-k^2) = \dfrac{1}{9-k^2}$; $y = \dfrac{\begin{vmatrix} 9 & 1 \\ k & 0 \end{vmatrix}}{9-k^2} = \dfrac{-k}{9-k^2}$

$$y = \frac{\begin{vmatrix} 9 & 1 \\ k & 0 \end{vmatrix}}{9-k^2} = \frac{-k}{9-k^2}$$

(c) $$A = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 0 & 4 \\ -1 & 4 & 2 \end{pmatrix} \xrightarrow[1R_1+R_3]{1R_1+R_2} \begin{pmatrix} 1 & 2 & -2 \\ 0 & 2 & 2 \\ 0 & 6 & 0 \end{pmatrix}$$

$$\xrightarrow{-3R_2+R_3} \begin{pmatrix} 1 & 2 & -2 \\ 0 & 2 & 2 \\ 6 & 0 & -6 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 3 & 1 \end{pmatrix}$$

Claim $A = LU = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -2 \\ 0 & 2 & 2 \\ 0 & 0 & -6 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 2 & -2 \\ -1 & 0 & 4 \\ -1 & 4 & 2 \end{pmatrix} \checkmark$$

(d) LU decomposition is good if you have to solve a systems w. the same coeff. matrix several times.

9. $M = \begin{pmatrix} 7 & 2 \\ 2 & 4 \end{pmatrix}$

(a) $M^T = \begin{pmatrix} 7 & 2 \\ 2 & 4 \end{pmatrix} = M \Rightarrow M$ is symmetric. So M is diagonalizable.

(b) $\Delta(\lambda) = \det(M - \lambda I) = \begin{vmatrix} 7-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix}$

$tr(M) = 11$, $\det(M) = 28 - 4 = 24$

$\Rightarrow \Delta(\lambda) = \lambda^2 - 11\lambda + 24 = 0$

$(\lambda - 3)(\lambda - 8) = 0$

$\Rightarrow$ e-vals of M are $\lambda_1 = 3$, $\lambda_2 = 8$

$\therefore \lambda_1 = 3 \quad X_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \left(\begin{array}{cc|c} 4 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right)$

$\xrightarrow{rref} \left(\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 = -\frac{1}{2}x_2 \\ x_2 = x_2 \text{ free} \end{array}$

$\therefore \lambda_1 = 3$, $X_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$;

$\lambda_2 = 8$, $X_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \left(\begin{array}{cc|c} -1 & 2 & 0 \\ 2 & -4 & 0 \end{array}\right)$

$\xrightarrow{rref} \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 = 2x_2 \\ x_2 = x_2 \text{ free} \end{array}$

#4 cont'd:

$$\lambda_2 = 8, \; X_2' = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

(d) $P = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$ $\left(\det P = -1-4 = -5 \neq 0 \;\therefore P^{-1} \text{ exists}\right)$

$$P^{-1}MP = D, \; D = \begin{pmatrix} 3 & 0 \\ 0 & 8 \end{pmatrix}$$

$$P^{-1} = -\frac{1}{5}\begin{pmatrix} 1 & -2 \\ -2 & -1 \end{pmatrix} \Rightarrow$$

$$P^{-1}MP = -\frac{1}{5}\begin{pmatrix} 1 & -2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 7 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$= -\frac{1}{5}\begin{pmatrix} 3 & -6 \\ -16 & -8 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$= -\frac{1}{5}\begin{pmatrix} -15 & 0 \\ 0 & -40 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 8 \end{pmatrix}$$

So this checks that P really does diagonalize M.

10. (a) $A\mathbf{X} = \begin{pmatrix}2&0&0\\0&2&-2\\0&-1&3\end{pmatrix}\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}0\\4\\-4\end{pmatrix} = 4\begin{pmatrix}0\\1\\-1\end{pmatrix}$

$\Rightarrow$ $\mathbf{X}$ is an e-vec. of A w. $\lambda = 4$ as its corresponding e-val

(b) $A^T = \begin{pmatrix}2&0&0\\0&2&-1\\0&-2&3\end{pmatrix} \neq A$ so A is not symmetric. It can still be diagonalizable. We need to find all e-vals of A: $\det(A-\lambda I)=0$ (solutions for λ)

$$\Delta(\lambda) = \begin{vmatrix}2-\lambda&0&0\\0&2-\lambda&-1\\0&-2&3-\lambda\end{vmatrix} = -(\lambda-2)(\lambda^2-5\lambda+4)$$

$= -(\lambda-2)(\lambda-1)(\lambda-4) \Rightarrow$

$\lambda_1=2, \lambda_2=1, \lambda_3=4$ are 3 distinct e-vals of A

So A is diagonalizable - b/c it has then 3 l.i. e-vals

(c) $\text{rref}(A) = \text{rref}\begin{pmatrix}2&0&0\\0&2&-2\\0&-1&3\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \Rightarrow$

$\text{rank}(A) = \#$ of nonzero rows in rref $\Rightarrow \text{rank}(A)=3$