Final Exam, SM331H (Popovici), Wednesday, Dec 13, 13:30

Name: ____

The last page of the exam is for stating the theorems you use while completing the problems. Label them as Theorem A, B, C, and refer to them in your problems by the letter you designated. The only exception is for the (ϵ, δ), the open set or the sequence descriptions of continuity, which you do not have to restate; simply refer to them as mentioned.

1. a. Give example of a function $g : \mathbf{R} \to \mathbf{R}$ such that $g(2) = g(6) = 0$, and such that 0 is an interior point of its range. Find

1a i. $R(g)$.

1a ii. $R(g)^o$.

1a iii. $g^{-1}(g[5, \infty))$.

1. b. Let $f : \mathbf{R} \to \mathbf{R}$ and let $W \subset \mathbf{R}$. One of the following statements is always true. Identify the statement by circling it and then prove it:

$$f^{-1}(f(W)) \subset W \quad f^{-1}(f(W)) \supset W.$$

2. a. Define $f : [0, \infty) \to \mathbf{R}$ as $f(x) = \sqrt{x}$. Use the (ϵ, δ) descriptions of continuity to prove that f is continuous at 0.

2. b. Define $g : \mathbf{R} \to \mathbf{R}$ as $g(x) = max\{0, x\} = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x \end{cases}$

Prove that g is continuous at $x = 0$.

Remark: one can show that g is continuous. You may use that fact without completing the proof for non-zero points for the rest of problem 2, if helpful.

2.c. Let $w : \mathbf{R} \to \mathbf{R}$ be continuous. Define $h : \mathbf{R} \to \mathbf{R}$ as $h(x) = \begin{cases} w(x) & \text{if } w(x) \geq 0 \\ 0 & \text{if } w(x) < 0 \end{cases}$

2.c.i. Prove that the set $\{x, \ \ w(x) > 0\}$ is open.

2.c.ii. Prove that h is continuous.

3. Let $f : [0, 10] \to \mathbf{R}$ be defined as $f(x) = \begin{cases} x & \text{if } x \in \mathbf{Q} \\ 8 - x & \text{if } x \notin \mathbf{Q} \end{cases}$

3.a. Prove that $f \circ f(x) = x$ for all $x \in [0, 10]$.

3.b.i. Show that f is not continuous at $\sqrt{2}$.

3.b.ii. Identify a point in $[0, 10]$ where f is continuous, and show it is continuous there.

4.a. For the set $\mathbf{Q}$ find (and prove your result):

4.a.i. $\mathbf{Q}^o$.

4.a.ii. $\partial\mathbf{Q}$.

4.a.iii. $\mathbf{Q}'$.

4.a.iv. sets A, B that are a disconnection of $\mathbf{Q}$.

4.b. Let A, B be subsets of $\mathbf{R}$. The following result is true:

$$A^o \bigcap B^o = (A \bigcap B)^o.$$

Prove the inclusion $A^o \bigcap B^o \subset (A \bigcap B)^o$.

4.c. Let A be a bounded set such that $\sup A \notin A$. Prove that $\sup A \in \partial A$.

5.a. Let $(x_n)_{n \in N}$ be a sequence in $\mathbf{R}$, convergent to a positive number x. Prove that $x_n > 0$ for all but finitely many n.

5.c. Consider the sequence $x_1 = \sqrt{3},\ x_2 = \sqrt{3\sqrt{3}},\ x_3 = \sqrt{3\sqrt{3\sqrt{3}}}, \ldots$.

5.c.i. Find a recurrence formula for the sequence.

5.c.ii. Prove that x_n is a monotone sequence in $[1, 3]$ (hint: induction).

5.c.iii. Find its limit.

6.a. Determine whether or not the infinite series below converge or diverge. If convergent find the sum.

6.a.i . $\sum_{n\in\mathbf{N}} \frac{1-2^n}{n+2^n}$.

6.a.ii . $\sum_{n=2}^{\infty} \frac{2^{n-1}}{6^n}$.

6. b. Let x_n be a sequence in $\mathbf{R}$ such that for any $n < m \in \mathbf{N}$, the following holds:

$$|x_n + x_{n+1} + x_{n+2} + \ldots + x_{m-1} + x_m| \leq \frac{1}{n} - \frac{1}{m}.$$

Prove that the series $\sum x_n$ converges.