Final Exam, SM331H, Dec 11, 2013 (6 pages). NAME:

PART 1. (20 points) 1. Provide definitions for the following terms. Be precise.

a. The supremum of a set A.

b. A boundary point x for of a set A.

c. A compact set K.

d. A closed set F.

e. The uniform (or sup) norm for a bounded function $f : D \to \mathbf{R}$

2. Fill out the following table (no explanations needed):

Set A	open?	closed?	compact?	boundary	interior	cluster points
[0,1)						
[0,1]						
$[0,\infty)$						
$[0,1)\bigcap \mathbf{Q}$						
$\mathbf{Z}$						
$\{(-1)^n/n : n \in \mathbf{N}\}$						

PART 2 (20 points). State the following theorems:

a. Heine-Borel Theorem

b. A theorem regarding the intersection of open sets.

c. Limit comparison test for a series.

d. Cauchy criterion for uniform convergence of functions.

e. A theorem regarding convergence of increasing sequences in $\mathbf{R}$.

f. Bolzano-Weirstrass theorem.

PART 3 (20 points): Prove the following theorems:

a. If f and g are functions $f, g : D \to \mathbf{R}$ with bounded ranges, then

$$\sup\{f(x) + g(x) : \ x \in D\} \leq \sup\{f(x) : \ x \in D\} + \sup\{g(x) : \ x \in D\}.$$

b. If (x_n) is a Cauchy sequence in $\mathbf{R}$, then there exists $x \in \mathbf{R}$ such that (x_n) converges to x.

PART 4 (42 points)

1. Give examples of (no explanations needed):

a. A sequence (x_n) with $x_n < 4$ for all $n \geq 1$ and $\liminf x_n = 4$.

b. A series Σx_n for which the application of ratio test leads to the conclusion that Σx_n converges.

c. A set A in $\mathbf{R}$ such that $\partial A = \{1, 2, 3\}$ and $A \neq \{1, 2, 3\}$.

2. Let $A = \{\frac{5}{1}, \frac{5}{2}, \frac{5}{3}, \frac{5}{4}, \dots \frac{5}{n} \dots\}$, let P be the set containing products of elements of A and elements of the interval $(2, 6)$, meaning: $P = \{ab |\ a \in A,\ 2 < b < 6\}$.

a. Fill in the blanks (no explanations needed):

$\sup A =$ $\qquad\qquad$ $\inf A =$ $\qquad\qquad$ $\sup P =$ $\qquad\qquad$ $\inf P =$ $\qquad\qquad$.

b. Prove that P is open.

3. Assume that $g, f_n : [0, \infty) \to [0, 3]$ are functions such that $f_n \to 0$ uniformly on $[0, 5]$.

Prove that the composition function $f_n \circ g$ converges to zero uniformly on $[0, \infty)$.

4. Let $f_n, f : [0,25] \to \mathbf{R}$ be defined as $f(x) = \sqrt{x}$ and $f_n(x) = \sqrt{x + 1/n}$ for all $x \in [0,25]$. Show that f_n converges uniformly to f on $[0,25]$.

5. CHOOSE ONE of two options:

Option 1: Let $X = \{(x,y)| \ x^2 + \frac{y^2}{x^2+2} \leq 4\}$. Prove X is compact.

Option 2: Let $(x_n) \subset \mathbf{R}$ be a sequence convergent to x; let $A = \{x\} \bigcup \{x_n \mid n \in \mathbf{N}\}$. Prove A is compact.

6. Let x_n, be a sequence in $[-10, 7]$. Show that if $\sum_{n=1}^{\infty} y_n$ converges absolutely, then $\sum_{n=1}^{\infty} x_n y_n$ converges.

7. Briefly explain whether the series bellow converge or diverge:

$$a.\ \Sigma \frac{1}{(\ln n + 5)^4} \qquad b.\ \Sigma \frac{1}{(n + \ln 5)^4} \qquad c.\ \Sigma \frac{(-1)^n}{n + \ln 5}.$$