Part I: No Calculators. State THREE of the following:

1. Taylor’s Theorem
2. Weirstrass M-test
3. Mean Value Theorem (derivative version)
4. Mean Value Theorem (integral version)
5. Definition of the radius of convergence for a power series
Part II: No Calculators. State ALL, prove ONE of the following:

1. A theorem connecting Riemann integrability and continuity.
2. A theorem connecting the convergence of a sequence of functions and the differentiability of the limit function. (partial proof)
3. A theorem connecting the convergence of a sequence of functions and the continuity of the limit function.
4. The Fundamental Theorem of Calculus
Part III. Give examples of the following; briefly explain why they work:

1. A function that is integrable but not differentiable.

2. A power series that converges for $x = 2$ and diverges for $x = 6$.

3. A non-integrable function.

4. A number $\alpha$ such that the series below has radius of convergence $R = 1/3$.

$$\frac{\alpha}{5} x^4 + \frac{\alpha^2}{9} x^8 + \frac{\alpha^3}{13} x^{12} + \frac{\alpha^4}{17} x^{16} + \frac{\alpha^5}{21} x^{20} + \frac{\alpha^6}{25} x^{24} + \ldots$$

5. A sequence $a_n$ that converges to zero, yet $\sum a_n$ diverges.

6. A number $p$ such that $\int_0^1 x^p d(x^3) = 1/5$
Part IV: Solve

1. Let \( f_n(x) = \frac{\cos(nx)}{n^2 + x} \) be functions defined on \([0, 2\pi]\).

   a. Show that \( \sum f_n \) is uniformly convergent on \([0, 2\pi]\). Denote the sum by \( f(x) \).

   b. Apply Integration by Parts Theorem to rewrite the integral of \( f_n \) as indicated/started:

   \[
   \int_{0}^{2\pi} f_n \, dx = \int_{0}^{2\pi} \frac{1}{n^2 + x} d \left( \frac{1}{n} \sin(nx) \right) = 
   
   \]

   Use the simplified/new integral to prove that \( |\int_{0}^{2\pi} f \, dx| \leq 2\pi \sum \frac{1}{n^2} \).
2. Let \( f : [0, 1] \to \mathbb{R} \) be defined as \( f(x) = \sin \frac{2\pi}{1-x} \) for \( x \in [0, 1) \), and \( f(1) = 0 \).

a. Circle all the properties that apply for the function \( f \). Cross out the failed properties. Briefly justify your first and last answers.

- bounded
- continuous
- differentiable on \([0, 6/7]\)
- integrable on \([0, 1]\)

b. Circle all the subintervals of \([0, 1]\) for which Rolle’s theorem applies (for the function \( f \)). Cross out the intervals for which Rolle’s does not apply and briefly justify why it doesn’t:

\([0, 1/3] \quad [0, 2/3] \quad [0, 1]\).

For one interval ONLY find one point whose existence is guaranteed by Rolle’s theorem.

c. Define \( F(x) = \int_0^x \sin \frac{2\pi}{1-t} \, dt \). Find or explain why \( \lim_{x \to 1/3} \frac{F(x) - F(1/3)}{x - 1/3} \) does not exist.

d. Consider the sequences \((x_n), (y_n)\) defined by the following sums

\[
x_n = \frac{1}{n} \sum_{k=1}^{n} \sin\left(\frac{2k\pi}{n}\right) \quad \text{and} \quad y_n = \frac{1}{n} \sum_{k=0}^{n-1} \sin\left(\frac{2\pi n}{n-k}\right) = \frac{1}{n} \sum_{k=1}^{n} \sin\left(\frac{2\pi k}{n}\right).
\]

Determine whether they are convergent or divergent. If convergent, what is their limit? You may leave the answer in terms of \( f \) or \( F \).
3. Identify whether the series in \( \mathbb{R} \) given below are convergent or divergent. Briefly explain your answer.

(a) \( \sum (-1)^n e^{1/n} \)
(b) \( \sum (-1)^n \frac{1}{n + \log n} \)
(c) \( \sum \frac{n}{4^n} \)

4. Let \( f(x) = e^x + e^{-x} \). (a) Find a formula for \( T_{2n} \), the Taylor polynomial of degree \( 2n \).

(b) Use Taylor Theorem to prove that for all \( x \geq 0 \)

\[
2 + x^2 + \frac{x^4}{12} \leq e^x + e^{-x}
\]
5. You may choose ONE of the following for problem 2:

**OPTION 1 (OLD):** Let \( f_n : [0, 4] \to [0, \infty) \) be defined as \( f_n(x) = \sqrt{x + \frac{1}{n}} \) for all \( x \in [0, 4] \). Note that the functions \( f_n \) are differentiable on \([0, 4] \).

a. Find the function \( f \), the (pointwise) limit of the sequence \((f_n)\).

b. Is \( f_n \) uniformly convergent on \([0, 4]\)? Prove your answer.

c. Is \( f'_n \) uniformly convergent on \([0, 4]\)? Prove your answer.

**OPTION 2 (NEW):** Let \( D_n \) denote the set of rational numbers in the interval \((0, 1]\) whose reduced denominator is less or equal to \( n \). For example \( D_4 = \{1, 1/2, 1/3, 2/3, 1/4, 3/4\} \). Use these sets to define \( f_n : [0, 1] \to [0, \infty) \) as

\[
f_n(x) = \begin{cases} 
\pi & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\
\pi & \text{if } x \in \mathbb{Q} \setminus D_n \\
0 & \text{if } x \in D_n.
\end{cases}
\]

a. Sketch (separately) the graphs of \( f_3, f_4, f_5 \).

b. Find the function \( f \), the (pointwise) limit of the sequence \((f_n)\). Briefly justify your answer.

c. What can be said about the integrability of the functions \( f_n \), of \( f \) on \([0, 1]\)? Briefly justify your answer.