Final Exam, SM332H, May 6, 2011 Name:

Part I: No Calculators. State THREE of the following:

1. Taylor's Theorem
2. Weirstrass M-test
3. Mean Value Theorem (derivative version)
4. Mean Value Theorem (integral version)
5. Definition of the radius of convergence for a power series

Part II : No Calculators. State ALL, prove ONE of the following:

1. A theorem connecting Riemann integrability and continuity.

2. A theorem connecting the convergence of a sequence of functions and the differentiability of the limit function. (partial proof)

3. A theorem connecting the convergence of a sequence of functions and the continuity of the limit function.

4. The Fundamental Theorem of Calculus

Part III. Give examples of the following; briefly explain why they work:

1. A function that is integrable but not differentiable.

2. A power series that converges for $x = 2$ and diverges for $x = 6$.

3. A non-integrable function.

4. A number α such that the series below has radius of convergence $R = 1/3$.

$$\frac{\alpha}{5}x^4 + \frac{\alpha^2}{9}x^8 + \frac{\alpha^3}{13}x^{12} + \frac{\alpha^4}{17}x^{16} + \frac{\alpha^5}{21}x^{20} + \frac{\alpha^6}{25}x^{24} + \dots$$

5. A sequence a_n that converges to zero, yet $\sum a_n$ diverges.

6. A number p such that $\int_0^1 x^p d(x^3) = 1/5$

Part IV: Solve

1. Let $f_n(x) = \frac{\cos(nx)}{n^2+x}$ be functions defined on $[0, 2\pi]$.

a. Show that $\sum f_n$ is uniformly convergent on $[0, 2\pi]$. Denote the sum by $f(x)$.

b. Apply Integration by Parts Theorem to rewrite the integral of f_n as indicated/started:

$$\int_0^{2\pi} f_n \, dx = \int_0^{2\pi} \frac{1}{n^2+x} d\left(\frac{1}{n}\sin(nx)\right) =$$

Use the simplified/new integral to prove that $|\int_0^{2\pi} f \, dx| \leq 2\pi \sum \frac{1}{n^5}$.

2. Let $f : [0,1] \to \mathbf{R}$ be defined as $f(x) = \sin \frac{2\pi}{1-x}$ for $x \in [0,1)$, and $f(1) = 0$.

a. Circle all the properties that apply for the function f. Cross out the failed properties. Briefly justify your first and last answers.

bounded   continuous   differentiable on $[0, 6/7]$ integrable on $[0, 1]$

b. Circle all the subintervals of $[0, 1]$ for which Rolle's theorem applies (for the function f). Cross out the intervals for which Rolle's does not apply and briefly justify why it doesn't: [0,1/3]   [0, 2/3], $[0, 1]$.

For one interval ONLY find one point whose existence is guaranteed by Rolle's theorem.

c. Define $F(x) = \int_0^x \sin \frac{2\pi}{1-t}\, dt$. Find or explain why $\lim_{x\to 1/3} \frac{F(x)-F(\frac{1}{3})}{x-\frac{1}{3}}$ does not exist.

d. Consider the sequences (x_n), (y_n) defined by the following sums

$$x_n = \frac{1}{n}\sum_{k=1}^{n} \sin(\frac{2k\pi}{n}) \quad \text{and} \quad y_n = \frac{1}{n}\sum_{k=0}^{n-1} \sin(\frac{2n\pi}{n-k}) = \frac{1}{n}\sum_{k=1}^{n} \sin(\frac{2n\pi}{k}).$$

Determine whether they are convergent or divergent. If convergent, what is their limit? You may leave the answer in terms of f or F.

3. Identify whether the series in $\mathbf{R}$ given below are convergent or divergent. Briefly explain your answer.

(a) $\sum (-1)^n e^{1/n}$

(b) $\sum (-1)^n \frac{1}{n+\log n}$

(c) $\sum \frac{n}{4^n}$

4. Let $f(x) = e^x + e^{-x}$. (a). Find a formula for T_{2n}, the Taylor polynomial of degree $2n$.

(b) Use Taylor Theorem to prove that for all $x \geq 0$

$$2 + x^2 + \frac{x^4}{12} \leq e^x + e^{-x}$$

5. You may choose ONE of the following for problem 2:

OPTION 1 (OLD): Let $f_n : [0, 4] \to [0, \infty)$ be defined as $f_n(x) = \sqrt{x + \frac{1}{n}}$ for all $x \in [0, 4]$. Note that the functions f_n are differentiable on $[0, 4]$.

a. Find the function f, the (poitwise) limit of the sequence (f_n).

b. Is f_n uniformly convergent on $[0, 4]$? Prove your answer.

c. Is f'_n uniformly convergent on $[0, 4]$? Prove your answer.

OPTION 2 (NEW): Let $\mathcal{D}_n$ denote the set of rational numbers in the interval $(0, 1]$ whose reduced denominator is less or equal to n. For example $\mathcal{D}_4 = \{1, 1/2, 1/3, 2/3, 1/4, 3/4\}$. Use these sets to define $f_n : [0, 1] \to [0, \infty)$ as

$$f_n(x) = \begin{cases} \pi & \text{if } x \in \mathbf{R} \setminus \mathbf{Q} \\ \pi & \text{if } x \in \mathbf{Q} \setminus \mathcal{D}_n \\ 0 & \text{if } x \in \mathcal{D}_n. \end{cases}$$

a. Sketch (separately) the graphs of f_3, f_4, f_5.

b. Find the function f, the (poitwise) limit of the sequence (f_n). Briefly justify your answer.

c. What can be said about the integrability of the functions f_n, of f on $[0, 1]$.? Briefly justify your answer.