Trigonometry

Trigonometric Functions

T1. $$\sin^2 x + \cos^2 x = 1$$

T2. $$\tan^2 x + 1 = \sec^2 x$$

T3. $$\cot^2 x + 1 = \csc^2 x$$

T4. $$\sin(x \pm y) = \sin x \, \cos y \pm \cos x \, \sin y$$

T5. $$\cos(x \pm y) = \cos x \, \cos y \mp \sin x \, \sin y$$

T6. $$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \, \tan y}$$

T7. $$\tan(\frac{x}{2}) = \frac{\sin x}{1 + \cos x}$$

T8. $$\sin(2x) = 2 \sin x \, \cos x$$

T9. $$\cos(2x) = \cos^2 x - \sin^2 x$$

T10. $$\sin^2 x = {}^1\!/_2(1 - \cos(2x))$$

T11. $$\cos^2 x = {}^1\!/_2(1 + \cos(2x))$$

T12. $$\sin x \, \sin y = {}^1\!/_2\big(\cos(x - y) - \cos(x + y)\big)$$

T13. $$\cos x \, \cos y = {}^1\!/_2\big(\cos(x - y) + \cos(x + y)\big)$$

T14. $$\sin x \, \cos y = {}^1\!/_2\big(\sin(x - y) + \sin(x + y)\big)$$

T15. $$c_1 \cos(\omega t) + c_2 \sin(\omega t) = A\ \sin(\omega t + \phi),$$

$$\text{where } A = \sqrt{c_1^2 + c_2^2}, \quad \phi = 2 \arctan \frac{c_1}{c_2 + A}$$

Hyperbolic Functions

T16. $$\cosh x = \frac{e^x + e^{-x}}{2}$$

T17. $$\sinh x = \frac{e^x - e^{-x}}{2}$$

T18. $$\cosh^2 x - \sinh^2 x = 1$$

T19. $$\tanh^2 x + \operatorname{sech}^2 x = 1$$

T20. $$\coth^2 x - \operatorname{csch}^2 x = 1$$

T21. $$\sinh(x \pm y) = \sinh x\ \cosh y \pm \cosh x\ \sinh y$$

T22. $$\cosh(x \pm y) = \cosh x\ \cosh y \pm \sinh x\ \sinh y$$

T23. $$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x\ \tanh y}$$

T24. $$\sinh(2x) = 2 \sinh x\ \cosh x$$

T25. $$\cosh(2x) = \cosh^2 x + \sinh^2 x$$

T26. $$\sinh x\ \sinh y = {}^1\!/\!_2\big(\cosh(x+y) - \cosh(x-y)\big)$$

T27. $$\cosh x\ \cosh y = {}^1\!/\!_2\big(\cosh(x+y) + \cosh(x-y)\big)$$

T28. $$\sinh x \cosh y = {}^1\!/\!{}_2\big(\sinh(x+y) + \sinh(x-y)\big)$$

Power Series

P1. $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \quad -\infty < x < \infty$$

P2. $$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \quad -\infty < x < \infty$$

P3. $$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots, \quad -\infty < x < \infty$$

P4. $$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

P5. $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots, \quad -1 < x < 1$$

P6. $$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots, \quad -\infty < x < \infty$$

P7. $$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots, \quad -\infty < x < \infty$$

P8. $$\tanh x = x - \frac{x^3}{3} + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \cdots, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

P9. $$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

P10. Taylor Series with remainder:

$$\begin{aligned} f(x) &= \textstyle\sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{N+1}(x), \quad \text{where} \\ R_{N+1}(x) &= \frac{f^{(N+1)}(\xi)}{(N+1)!}(x-a)^{N+1} \qquad \text{for some } \xi \text{ between } a \text{ and } x. \end{aligned}$$

Derivative formulas

1. **Product rule**: $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$
2. **Chain rule**:

$$f(g(x))' = f'(g(x))g'(x)$$
$$\frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt}$$

3. **Quotient rule**: $(\frac{f(x)}{g(x)})' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$
4. $\frac{d}{dx}(e^{cx}) = c \cdot e^{cx}$
5. $\frac{d}{dx}\sin(cx) = c \cdot \cos(cx)$
6. $\frac{d}{dx}\cos(cx) = -c \cdot \sin(cx)$

Algebra formulas

1. **Quadratic formula**: Roots of $ax^2 + bx + c = 0$ are: $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$
2. $a^2 - b^2 = (a-b)(a+b)$
3. $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ and $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
4. $a^n - b^n = (a-b)(a^{n-1} + ba^{n-2} + ... + b^{n-2}a + b^{n-1})$

Geometry formulas

1. **area of a triangle** of base b, height h: $A = bh/2$
2. **volume of a sphere** of radius r: $V = 4\pi r^3/3$
2. **surface area of a sphere** of radius r: $V = 4\pi r^2$

Table of Integrals

A constant of integration should be added to each formula. The letters a, b, m, and n denote constants; u and v denote functions of an independent variable such as x.

Standard Integrals

I1. $$\int u^n \, du = \frac{u^{n+1}}{n+1}, \quad n \neq -1$$

I2. $$\int \frac{du}{u} = \ln |u|$$

I3. $$\int e^u \, du = e^u$$

I4. $$\int a^u \, du = \frac{a^u}{\ln a}, \quad a > 0$$

I5. $$\int \cos u \, du = \sin u$$

I6. $$\int \sin u \, du = -\cos u$$

I7. $$\int \sec^2 u \, du = \tan u$$

I8. $$\int \csc^2 u \, du = -\cot u$$

I9. $$\int \sec u \tan u \, du = \sec u$$

I10. $$\int \csc u \cot u \, du = -\csc u$$

I11. $$\int \tan u \, du = -\ln |\cos u|$$

I12. $$\int \cot u \, du = \ln |\sin u|$$

I13. $$\int \sec u\, du = \ln|\sec u + \tan u|$$

I14. $$\int \csc u\, du = \ln|\csc u - \cot u|$$

I15. $$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right)$$

I16. $$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right)$$

I17. $$\int u\, dv = uv - \int v\, du$$

Integrals involving $au+b$

I18. $$\int (au+b)^n du = \frac{(au+b)^{n+1}}{(n+1)a}, \quad n \neq -1$$

I19. $$\int \frac{du}{au+b} = \frac{1}{a}\ln|au+b|$$

I20. $$\int \frac{u\, du}{au+b} = \frac{u}{a} - \frac{b}{a^2}\ln|au+b|$$

I21. $$\int \frac{u\, du}{(au+b)^2} = \frac{b}{a^2(au+b)} + \frac{1}{a^2}\ln|au+b|$$

I22. $$\int \frac{du}{u(au+b)} = \frac{1}{b}\ln\left|\frac{u}{au+b}\right|$$

I23. $$\int u\sqrt{au+b}\, du = \frac{2(3au-2b)}{15a^2}(au+b)^{3/2}$$

I24. $$\int \frac{u\, du}{\sqrt{au+b}} = \frac{2(au-2b)}{3a^2}\sqrt{au+b}$$

I25. $$\int u^2\sqrt{au+b}\, du = \frac{2}{105a^3}\left(8b^2 - 12abu + 15a^2u^2\right)(au+b)^{3/2}$$

I26. $$\int \frac{u^2\, du}{\sqrt{au+b}} = \frac{2}{15a^3}\left(8b^2 - 4abu + 3a^2u^2\right)\sqrt{au+b}$$

Integrals involving $u^2 \pm a^2$

I27. $$\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln\left|\frac{u-a}{u+a}\right|$$

I28. $$\int \frac{u\,du}{u^2 \pm a^2} = {}^1\!/_2 \ln\left|u^2 \pm a^2\right|$$

I29. $$\int \frac{u^2\,du}{u^2 - a^2} = u + \frac{a}{2} \ln\left|\frac{u-a}{u+a}\right|$$

I30. $$\int \frac{u^2\,du}{u^2 + a^2} = u - a \arctan\left(\frac{u}{a}\right)$$

I31. $$\int \frac{du}{u(u^2 \pm a^2)} = \pm\frac{1}{2a^2} \ln\left|\frac{u^2}{u^2 \pm a^2}\right|$$

Integrals involving $\sqrt{u^2 \pm a^2}$

I32. $$\int \frac{u\,du}{\sqrt{u^2 \pm a^2}} = \sqrt{u^2 \pm a^2}$$

I33. $$\int u\sqrt{u^2 \pm a^2}\,du = \frac{1}{3}\left(u^2 \pm a^2\right)^{3/2}$$

I34. $$\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln\left|u + \sqrt{u^2 \pm a^2}\,\right|$$

I35. $$\int \frac{u^2\,du}{\sqrt{u^2 \pm a^2}} = \frac{u}{2}\sqrt{u^2 \pm a^2} \mp \frac{a^2}{2} \ln\left|u + \sqrt{u^2 \pm a^2}\,\right|$$

I36. $$\int \frac{du}{u\sqrt{u^2 + a^2}} = \frac{1}{a} \ln\left|\frac{u}{a + \sqrt{u^2 + a^2}}\right|$$

I37. $$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec}\left(\frac{u}{a}\right)$$

I38. $$\int \frac{du}{u^2\sqrt{u^2 \pm a^2}} = \mp\frac{\sqrt{u^2 \pm a^2}}{a^2 u}$$

I39. $$\int \sqrt{u^2 \pm a^2}\,du = \frac{u}{2}\sqrt{u^2 \pm a^2} \pm \frac{a^2}{2} \ln\left|u + \sqrt{u^2 \pm a^2}\,\right|$$

I40. $$\int u^2\sqrt{u^2 \pm a^2}\,du = \frac{u}{4}\left(u^2 \pm a^2\right)^{3/2} \mp \frac{a^2 u}{8}\sqrt{u^2 \pm a^2} - \frac{a^4}{8}\ln\left|u + \sqrt{u^2 \pm a^2}\,\right|$$

I41. $$\int \frac{\sqrt{u^2 + a^2}}{u}\,du = \sqrt{u^2 + a^2} - a\ln\left|\frac{a + \sqrt{u^2 + a^2}}{u}\right|$$

I42. $$\int \frac{\sqrt{u^2 - a^2}}{u}\,du = \sqrt{u^2 - a^2} - a\,\mathrm{arcsec}\left(\frac{u}{a}\right)$$

I43. $$\int \frac{\sqrt{u^2 \pm a^2}}{u^2}\,du = -\frac{\sqrt{u^2 \pm a^2}}{u} + \ln\left|u + \sqrt{u^2 \pm a^2}\,\right|$$

Integrals involving $\sqrt{a^2 - u^2}$

I44. $$\int \sqrt{a^2 - u^2}\,du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\arcsin\left(\frac{u}{a}\right)$$

I45. $$\int \frac{u\,du}{\sqrt{a^2 - u^2}} = -\sqrt{a^2 - u^2}$$

I46. $$\int u\sqrt{a^2 - u^2}\,du = -\frac{1}{3}\left(a^2 - u^2\right)^{3/2}$$

I47. $$\int \frac{\sqrt{a^2 - u^2}}{u}\,du = \sqrt{a^2 - u^2} - a\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right|$$

I48. $$\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a}\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right|$$

I49. $$\int u^2\sqrt{a^2 - u^2}\,du = -\frac{u}{4}\left(a^2 - u^2\right)^{3/2} + \frac{a^2 u}{8}\sqrt{a^2 - u^2} + \frac{a^4}{8}\arcsin\left(\frac{u}{a}\right)$$

I50. $$\int \frac{\sqrt{a^2 - u^2}}{u^2}\,du = -\frac{\sqrt{a^2 - u^2}}{u} - \arcsin\left(\frac{u}{a}\right)$$

I51. $$\int \frac{u^2\,du}{\sqrt{a^2 - u^2}} = -\frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\arcsin\left(\frac{u}{a}\right)$$

I52. $$\int \frac{du}{u^2\sqrt{a^2 - u^2}} = -\frac{\sqrt{a^2 - u^2}}{a^2 u}$$

Integrals involving trigonometric functions

I53. $$\int \sin^2(au)\,du = \frac{u}{2} - \frac{\sin(2au)}{4a}$$

I54. $$\int \cos^2(au)\,du = \frac{u}{2} + \frac{\sin(2au)}{4a}$$

I55. $$\int \sin^3(au)\,du = \frac{1}{a}\left(\frac{\cos^3(au)}{3} - \cos(au)\right)$$

I56. $$\int \cos^3(au)\,du = \frac{1}{a}\left(\sin(au) - \frac{\sin^3(au)}{3}\right)$$

I57. $$\int \sin^2(au)\,\cos^2(au)\,du = \frac{u}{8} - \frac{1}{32a}\sin(4au)$$

I58. $$\int \tan^2(au)\,du = \frac{1}{a}\tan(au) - u$$

I59. $$\int \cot^2(au)\,du = -\frac{1}{a}\cot(au) - u$$

I60. $$\int \sec^3(au)\,du = \frac{1}{2a}\sec(au)\tan(au) + \frac{1}{2a}\ln|\,\sec(au) + \tan(au)\,|$$

I61. $$\int \csc^3(au)\,du = -\frac{1}{2a}\csc(au)\cot(au) + \frac{1}{2a}\ln|\,\csc(au) - \cot(au)\,|$$

I62. $$\int u\sin(au)\,du = \frac{1}{a^2}\left(\sin(au) - au\cos(au)\right)$$

I63. $$\int u\cos(au)\,du = \frac{1}{a^2}\left(\cos(au) + au\sin(au)\right)$$

I64. $$\int u^2\sin(au)\,du = \frac{1}{a^3}\big(2au\sin(au) - (\,a^2u^2 - 2\,)\cos(au)\big)$$

I65. $$\int u^2\cos(au)\,du = \frac{1}{a^3}\big(2au\cos(au) + (\,a^2u^2 - 2\,)\sin(au)\big)$$

I66. $$\int \sin(au)\,\sin(bu)\,du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)}, \qquad a^2 \neq b^2$$

I67. $$\int \cos(au)\,\cos(bu)\,du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)}, \qquad a^2 \neq b^2$$

I68. $$\int \sin(au)\,\cos(bu)\,du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)}, \qquad a^2 \neq b^2$$

I69. $$\int \sin^n u \, du = -\frac{1}{n} \sin^{n-1} u \cos u + \frac{n-1}{n} \int \sin^{n-2} u \, du$$

Integrals involving hyperbolic functions

I70. $$\int \sinh(au) \, du = \frac{1}{a} \cosh(au)$$

I71. $$\int \sinh^2(au) \, du = \frac{1}{4a} \sinh(2au) - \frac{u}{2}$$

I72. $$\int \cosh(au) \, du = \frac{1}{a} \sinh(au)$$

I73. $$\int \cosh^2(au) \, du = \frac{u}{2} + \frac{1}{4a} \sinh(2au)$$

I74. $$\int \sinh(au) \, \cosh(bu) \, du = \frac{\cosh\left((a+b)\,u\right)}{2(a+b)} + \frac{\cosh\left((a-b)\,u\right)}{2(a-b)}$$

I75. $$\int \sinh(au) \, \cosh(au) \, du = \frac{1}{4a} \cosh(2au)$$

I76. $$\int \tanh u \, du = \ln(\cosh u)$$

I77. $$\int \operatorname{sech} u \, du = \arctan(\sinh u) = 2 \arctan\left(e^u\right)$$

Integrals involving exponential functions

I78. $$\int u e^{au} \, du = \frac{e^{au}}{a^2}(au - 1)$$

I79. $$\int u^2 e^{au} \, du = \frac{e^{au}}{a^3} \left(a^2 u^2 - 2au + 2 \right)$$

I80. $$\int u^n e^{au} \, du = \frac{1}{a} u^n e^{au} - \frac{n}{a} \int u^{n-1} e^{au} \, du$$

I81. $$\int e^{au} \sin(bu) \, du = \frac{e^{au}}{a^2 + b^2} \left(a \sin(bu) - b \cos(bu) \right)$$

I82. $$\int e^{au}\cos(bu)\,du = \frac{e^{au}}{a^2+b^2}\left(a\cos(bu)+b\sin(bu)\right)$$

Integrals involving inverse trigonometric functions

I83. $$\int \arcsin\left(\frac{u}{a}\right)\,du = u\arcsin\left(\frac{u}{a}\right)+\sqrt{a^2-u^2}$$

I84. $$\int \arccos\left(\frac{u}{a}\right)\,du = u\arccos\left(\frac{u}{a}\right)-\sqrt{a^2-u^2}$$

I85. $$\int \arctan\left(\frac{u}{a}\right)\,du = u\arctan\left(\frac{u}{a}\right)-\frac{a}{2}\ln\left(a^2+u^2\right)$$

Integrals involving inverse hyperbolic functions

I86. $$\int \operatorname{arcsinh}\left(\frac{u}{a}\right)\,du = u\operatorname{arcsinh}\left(\frac{u}{a}\right)-\sqrt{u^2+a^2}$$

I87.

$$\begin{aligned}\int \operatorname{arccosh}\left(\frac{u}{a}\right)\,du &= u\operatorname{arccosh}\left(\tfrac{u}{a}\right)-\sqrt{u^2-a^2} \qquad \operatorname{arccosh}\left(\tfrac{u}{a}\right)>0;\\ &= u\operatorname{arccosh}\left(\tfrac{u}{a}\right)+\sqrt{u^2-a^2} \qquad \operatorname{arccosh}\left(\tfrac{u}{a}\right)<0.\end{aligned}$$

I88. $$\int \operatorname{arctanh}\left(\frac{u}{a}\right)\,du = u\operatorname{arctanh}\left(\frac{u}{a}\right)+\frac{a}{2}\ln\left(a^2-u^2\right)$$

Integrals involving logarithm functions

I89. $$\int \ln u\,du = u(\ln u-1)$$

I90. $$\int u^n\ln u\,du = u^{n+1}\left[\frac{\ln u}{n+1}-\frac{1}{(n+1)^2}\right], \qquad n\neq -1$$

Wallis' Formulas

I91.

$$\int_0^{\pi/2} \sin^m x\,dx = \int_0^{\pi/2} \cos^m x\,dx = \frac{(m-1)(m-3)\ldots(2\text{ or }1)}{m(m-2)\ldots(3\text{ or }2)}k,$$

where $k = 1$ if m is odd and $k = \pi/2$ if m is even.

I92. $$\int_0^{\pi/2} \sin^m x\,\cos^n x\,dx = \frac{(m-1)(m-3)\ldots(2\text{ or }1)(n-1)(n-3)\ldots(2\text{ or }1)}{(m+n)(m+n-2)\ldots(2\text{ or }1)}\,k,$$

where $k = \pi/2$ if both m and n are even and $k = 1$ otherwise.

Gamma Function

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}\,dt, \qquad x > 0$$

$$\Gamma(x+1) = x\Gamma(x)$$

$$\Gamma\left(1/2\right) = \sqrt{\pi}$$

$$\Gamma(n+1) = n!, \qquad \text{if } n \text{ is a non-negative integer}$$

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\Big)$$

where

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx.$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ni\pi x/L}$$

where

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-ni\pi x/L}\,dx.$$

Bessel Functions

$\nu =$ arbitrary real number; $n =$ integer

1. Definition.
   $x^2y'' + xy' + (\lambda^2x^2 - \nu^2)y = 0$ has solutions

$$y = y_\nu(\lambda x) = c_1J_\nu(\lambda x) + c_2J_{-\nu}(\lambda x), \qquad (\nu \neq n),$$

where

$$J_\nu(t) = \sum_{m=0}^{\infty} \frac{(-1)^m\, t^{\nu+2m}}{2^{\nu+2m}\, m!\,\Gamma(\nu+m+1)}.$$

2. General Properties.

$$J_{-n}(t) = (-1)^nJ_n(t); \quad J_0(0) = 1; \quad J_n(0) = 0, \qquad (n \geq 1).$$

3. Identities.
   (a) $\dfrac{d}{dt}[t^\nu J_\nu(t)] = t^\nu J_{\nu-1}(t).$

   (b) $\dfrac{d}{dt}[t^{-\nu} J_\nu(t)] = -t^{-\nu} J_{\nu+1}(t); \qquad \dfrac{d}{dt}J_0(t) = -J_1(t).$

(c)

$$\begin{aligned}\frac{d}{dt}J_\nu(t) &= 1/2\,(J_{\nu-1}(t) - J_{\nu+1}(t)) \\ &= J_{\nu-1}(t) - \tfrac{\nu}{t}J_\nu(t) \\ &= \tfrac{\nu}{t}J_\nu(t) - J_{\nu+1}(t)\,.\end{aligned}$$

(d) Recurrence Relation.

$$J_{\nu+1}(t) = \frac{2\nu}{t}J_\nu(t) - J_{\nu-1}(t)$$

4. Orthogonality.
   Solutions $y_\nu(\lambda_0 x),\ y_\nu(\lambda_1 x),\ \ldots, y_\nu(\lambda_n x), \ldots$ of the differential system

$$x^2 y'' + xy' + (\lambda^2 x^2 - \nu^2)y = 0, \qquad x_1 \le x \le x_2$$

$$a_k y_\nu(\lambda x_k) - b_k\left(\frac{d}{dx}y_\nu(\lambda x)\right)\bigg|_{x=x_k} = 0, \qquad k = 1, 2$$

   have the orthogonality property:

$$\int_{x_1}^{x_2} x y_\nu(\lambda_n x) y_\nu(\lambda_m x)\,dx = 0, \qquad (m \ne n)$$

   and

$$\int_{x_1}^{x_2} x y_\nu^2(\lambda_m x)\,dx =$$

$$\frac{1}{2\lambda_m^2}\left[\left(\lambda_m^2 x^2 - \nu^2\right) y_\nu^2(\lambda_m x) + x^2\left(\frac{d}{dx}y_\nu\left(\lambda_m x\right)\right)^2\right]_{x_1}^{x^2}, \qquad (m = n)\,.$$

5. Integrals.

(a) $\displaystyle\int t^\nu J_{\nu-1}(t)\,dt = t^\nu J_\nu(t) + C$

(b) $\displaystyle\int t^{-\nu} J_{\nu+1}(t)\,dt = -t^{-\nu} J_\nu(t) + C$

(c) $\displaystyle\int t\,J_0(t)\,dt = tJ_1(t) + C$

(d) $\displaystyle\int t^3 J_0(t)\,dt = (t^3 - 4t)J_1(t) + 2t^2 J_0(t) + C$

(e) $\displaystyle\int t^2 J_1(t)\,dt = 2tJ_1(t) - t^2 J_0(t) + C$

(f) $\displaystyle\int t^4 J_1(t)\,dt = (4t^3 - 16t)J_1(t) - (t^4 - 8t^2)J_0(t) + C$

Zeros and Associated Values of Bessel Functions				
α	$j_{0,\alpha}$	$J_0'(j_{0,\alpha})$	$j_{1,\alpha}$	$J_1'(j_{1,\alpha})$
1	2.40483	-0.519147	3.83170	-0.402760
2	5.52008	0.340265	7.01559	0.300116
3	8.65373	-0.271452	10.1735	-0.249705
4	11.7915	0.232460	13.3237	0.218359
5	14.9309	-0.206546	16.4706	-0.196465
6	18.0711	0.187729	19.6159	0.180063
7	21.2116	-0.173266	22.7601	-0.167185
8	24.3525	0.161702	25.9037	0.156725
9	27.4935	-0.152181	29.0468	-0.148011
10	30.6346	0.144166	32.1897	0.140606
11	33.7758	-0.137297	35.3323	-0.134211
12	36.9171	0.131325	38.4748	0.128617
13	40.0584	-0.126069	41.6171	-0.123668
14	43.1998	0.121399	44.7593	0.119250
15	46.3412	-0.117211	47.9015	-0.115274
16	49.4826	0.113429	51.0435	0.111670
17	52.6241	-0.109991	54.1856	-0.108385
18	55.7655	0.106848	57.3275	0.105374
19	58.9070	-0.103960	60.4695	-0.102601
20	62.0485	0.101293	63.6114	0.100035

Table of Laplace Transforms

	$f(t) \equiv \mathcal{L}^{-1}\{F(s)\}(t)$	$F(s) \equiv \mathcal{L}\{f(t)\}(s)$
L1.	$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)\, dt$
L2.	$1,\ H(t),\ U(t)$	$\frac{1}{s}$
L3.	$U(t-a)$	$\frac{e^{-as}}{s}$
L4.	$t^n \quad (n = 1, 2, 3, \ldots)$	$\frac{n!}{s^{n+1}}$
L5.	$t^a \quad (a > -1)$	$\frac{\Gamma(a+1)}{s^{a+1}}$
L6.	e^{at}	$\frac{1}{s-a}$
L7.	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
L8.	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
L9.	$f'(t)$	$sF(s) - f(0)$
L10.	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
L11.	$t^n f(t) \quad (n = 1, 2, 3, \ldots)$	$(-1)^n F^{(n)}(s)$
L12.	$e^{at} f(t)$	$F(s-a)$
L13.	$e^{at}\mathcal{L}^{-1}\{F(s+a)\}$	$F(s)$
L14.	$f(t+P) = f(t)$	$\frac{\int_0^P e^{-st} f(t)\, dt}{1 - e^{-sP}}$

Table of Laplace Transforms

	$f(t) \equiv \mathcal{L}^{-1}\left\{F(s)\right\}(t)$	$F(s) \equiv \mathcal{L}\left\{f(t)\right\}(s)$
L15.	$f(t)U(t-a)$	$e^{-as}\mathcal{L}\left\{f(t+a)\right\}$
L16.	$f(t-a)U(t-a)$	$e^{-as}F(s)$
L17.	$\int_0^t f(z)\,dz$	$\frac{F(s)}{s}$
L18.	$\int_0^t f(z)g(t-z)\,dz$	$F(s)G(s)$
L19.	$\frac{f(t)}{t}$	$\int_s^\infty F(z)\,dz$
L20.	$\frac{1}{a}\left(e^{at}-1\right)$	$\frac{1}{s\,(s-a)}$
L21.	$t^n e^{at},\ n=1,2,3,\ldots$	$\frac{n!}{(s-a)^{n+1}}$
L22.	$\frac{e^{at}-e^{bt}}{a-b}$	$\frac{1}{(s-a)(s-b)}$
L23.	$\frac{ae^{at}-be^{bt}}{a-b}$	$\frac{s}{(s-a)(s-b)}$
L24.	$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
L25.	$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
L26.	$\sin(\omega t)-\omega t\cos(\omega t)$	$\frac{2\omega^3}{(s^2+\omega^2)^2}$
L27.	$t\sin(\omega t)$	$\frac{2\omega s}{(s^2+\omega^2)^2}$
L28.	$\sin(\omega t)+\omega t\cos(\omega t)$	$\frac{2\omega s^2}{(s^2+\omega^2)^2}$

Table of Laplace Transforms

	$f(t) \equiv \mathcal{L}^{-1}\{F(s)\}(t)$	$F(s) \equiv \mathcal{L}\{f(t)\}(s)$
L29.	$\dfrac{b\sin(at) - a\sin(bt)}{ab(b^2 - a^2)}$	$\dfrac{1}{(s^2 + a^2)(s^2 + b^2)}$
L30.	$\dfrac{\cos(at) - \cos(bt)}{b^2 - a^2}$	$\dfrac{s}{(s^2 + a^2)(s^2 + b^2)}$
L31.	$\dfrac{a\sin(at) - b\sin(bt)}{a^2 - b^2}$	$\dfrac{s^2}{(s^2 + a^2)(s^2 + b^2)}$
L32.	$e^{-bt}\sin(\omega t)$	$\dfrac{\omega}{(s + b)^2 + \omega^2}$
L33.	$e^{-bt}\cos(\omega t)$	$\dfrac{s + b}{(s + b)^2 + \omega^2}$
L34.	$1 - \cos(\omega t)$	$\dfrac{\omega^2}{s(s^2 + \omega^2)}$
L35.	$\omega t - \sin(\omega t)$	$\dfrac{\omega^3}{s^2(s^2 + \omega^2)}$
L36.	$\delta(t - a)$	$e^{-sa} \quad a > 0,\ s > 0$
L37.	$\delta(t - a)f(t)$	$f(a)e^{-sa} \quad a > 0,\ s > 0$

Partial fraction decomposition (PFD) of $N(x)/D(x)$: Let $N(x)$ be a polynomial of lower degree than another polynomial $D(x)$.

1. Factor $D(x)$ into a product of distinct terms of the form $(ax + b)^r$ or $(ax^2 + bx + c)^s$, for some integers $r > 0$, $s > 0$. For each term $(ax + b)^r$ the PFD of $N(x)/D(x)$ contains a sum of terms of the form $\frac{A_1}{(ax+b)} + \cdots + \frac{A_r}{(ax+b)^r}$ for some contstants A_i, and for each term $(ax^2 + bx + c)^s$ the PFD of $N(x)/D(x)$ contains a sum of terms of the form $\frac{B_1 x + C_1}{(ax^2+bx+c)} + \cdots + \frac{B_s x + C_s}{(ax^2+bx+c)^s}$. for some constants B_i, C_j. $\frac{N(x)}{D(x)}$ is the sum of all these.

2. The next step is to solve for these A's, B's, C's occurring in the numerators. Cross multiply both sides by $D(x)$ and expand out the resulting polynomial identity for $N(x)$ in terms of the A's, B's, C's. Equating coefficients of powers of x on both sides gives rise to a linear system of equations for the A's, B's, C's which you can solve.

Table of Fourier Transforms

Complex Fourier Integral:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C_\omega e^{i\omega t} d\omega,$$

where $C_\omega = F(\omega) = \mathcal{F}\{f(t)\}(\omega) = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$. (In the table $a > 0$ everywhere it appears)

	$\mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$	$\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$
F1.	$f(t)$	$F(\omega)$
F2.	$e^{-at}H(t)$	$\frac{1}{a+i\omega}$
F3.	$e^{at}H(-t) + e^{-at}H(t)$	$\frac{2a}{a^2+\omega^2}$
F4.	$-e^{at}H(-t) + e^{-at}H(t)$	$\frac{-2i\omega}{a^2+\omega^2}$
F5.	$\begin{cases} c & \text{if } -k < t < 0 \\ b & \text{if } 0 < t < m \\ 0 & \text{otherwise} \end{cases}$	$\frac{1}{i\omega}\left\{(b-c) + ce^{i\omega k} - be^{-i\omega m}\right\}$
F6.	$k(H(t+a) - H(t-a))$	$\frac{2k}{\omega}\sin(a\omega)$
F7.	$f(t-t_0)$	$e^{-i\omega t_0}F(\omega)$
F8.	$e^{i\omega_0 t}f(t)$	$F(\omega - \omega_0)$
F9.	$f(kt)$	$\frac{1}{\|k\|}F(\frac{\omega}{k})$
F10.	$f(-t)$	$F(-\omega)$
F11.	$F(t)$	$2\pi f(-\omega)$

Table of Fourier Transforms

	$\mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$	$\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$
F12.	$e^{-a\|t\|}$	$\frac{2a}{a^2+\omega^2}$
F13.	e^{-at^2}	$\sqrt{\frac{\pi}{a}}e^{-\frac{\omega^2}{4a}}$
F14.	$\frac{1}{a^2+t^2}$	$\frac{\pi}{a}e^{-a\|\omega\|}$
F15.	$f^{(n)}(t)\ (n = 0, 1, 2, 3, ...)$	$(i\omega)^n F(\omega)$
F16.	$t^n f(t)\ (n = 0, 1, 2, 3, ...)$	$i^n F^{(n)}(\omega)$
F17.	$\int_{-\infty}^{\infty} f(u)g(t-u)du$	$F(\omega)G(\omega)$
F18.	$f(t)g(t)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(x)G(\omega - x)dx$
F19.	$\delta(t)$	1

Last modified 11-15-2006 by Prof A. M. Gaglione and Prof. W. D. Joyner.