Faculty Profile: Asst. Prof. Douglas S. Altner

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Problem of the Month

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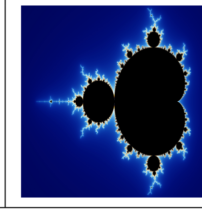

Countable vs. Uncountable Infinity

by Asst. Prof. Vrej Zarikian (`zarikian@usna.edu`)

One of the first courses USNA math majors take is SM291. The official title is *Fundamentals of Mathematics*, but it is commonly referred to as "fun math". Perhaps the most intriguing topic in SM291 is that of *cardinality*. According to Wikipedia,

> In mathematics, the cardinality of a set is a measure of the "number of elements in the set".

Why the quotes? Given a set such as $S = \{2, 4, 6, 8, 10\}$, the number of elements in S seems pretty unambiguous, five in this case. The problem is that many interesting sets in mathematics are *infinite* (= not finite). For example, the set of all natural numbers, $\mathbb{N}$, the set of all integers, $\mathbb{Z}$, the set of all rational numbers, $\mathbb{Q}$, and the set of all real numbers, $\mathbb{R}$. Since all of these sets are infinite, it is tempting to say that they all have the same cardinality, "infinity". On the other hand, $\mathbb{N}$ is a proper subset of $\mathbb{Z}$, because $\mathbb{Z}$ contains zero and negative whole numbers. Likewise, $\mathbb{Z}$ is a proper subset of $\mathbb{Q}$, because $\mathbb{Q}$ contains fractions such as $1/2$. Finally, $\mathbb{Q}$ is a proper subset of $\mathbb{R}$, because $\mathbb{R}$ contains irrational numbers such as $\sqrt{2}$ and π. That seems to suggest that there are different "levels" of infinity, and that $\mathbb{R}$ is "more infinite" than $\mathbb{Q}$, which is "more infinite" than $\mathbb{Z}$, which is "more infinite" than $\mathbb{N}$. The truth lies somewhere in between. There are different "levels" of infinity. However, $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are "equally infinite", whereas $\mathbb{R}$ is "more infinite". To understand this statement, we must consider the precise mathematical definitions.

Mathematicians say that an infinite set S is *countably infinite* if its elements can be listed one after the other, with no omissions. The quintessential example of a countably infinite set is $\mathbb{N}$, the set of all natural numbers.

†On the cover: The USNA seal (left) and the Mandelbrot set (right).

Here are its elements, listed one after the other, with no omissions:

$$1, 2, 3, 4, 5, ...$$

Another example of a countably infinite set is $\mathbb{Z}$, the set of all integers. Here are its elements, listed one after the other, with no omissions:

$$0, 1, -1, 2, -2, 3, -3, ...$$

Surprisingly, the set of all rational numbers, $\mathbb{Q}$, is also countably infinite. Here are its elements, listed one after the other, with no omissions:

$$0, 1, -1, 2, -2, \frac{1}{2}, -\frac{1}{2}, 3, -3, \frac{3}{2}, -\frac{3}{2}, \frac{1}{3}, -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}, ...$$

Do you see the pattern? The first three terms are all the fractions that have a numerator from the set $\{0, \pm 1\}$ and a denominator from the set $\{1\}$, the next four terms are all the fractions that have a numerator from the set $\{0, \pm 1, \pm 2\}$ and a denominator from the set $\{1, 2\}$, minus any repetitions. The next eight terms are all the fractions that have a numerator from the set $\{0, \pm 1, \pm 2, \pm 3\}$ and a denominator from the set $\{1, 2, 3\}$, minus any repetitions. Test your understanding by producing the next eight terms.[‡]

Mathematicians say that an infinite set S is *uncountably infinite* if it is not countably infinite, i.e. if its elements can't be listed one after the other, with no omissions. The German mathematician Georg Cantor (1845-1918) proved that the set of all real numbers, $\mathbb{R}$, is uncountably infinite. Using his famous *diagonalization argument*, we will demonstrate the slightly simpler fact that the set of all *binary sequences* is uncountably infinite. A binary sequence is nothing more than an unending string of 0s and 1s. For example,

$$10101010101010101010...$$

is a binary sequence. Now consider the following list of binary sequences (where for some reason we have underlined the first term of the first sequence, the second term of the second sequence, the third term of the third sequence, etc.):

$$\begin{gathered}
\underline{1}1111111111111111111...\\
0\underline{0}000000000000000000...\\
10\underline{1}01010101010101010...\\
010\underline{1}0101010101010101...\\
1100\underline{1}100110011001100...\\
00110\underline{0}11001100110011...\\
\vdots
\end{gathered}$$

[‡]Answer: $4, -4, \frac{4}{3}, -\frac{4}{3}, \frac{1}{4}, -\frac{1}{4}, \frac{3}{4}, -\frac{3}{4}$.

We claim that this list contains an omission. Indeed, consider the binary sequence whose terms are the opposites of the underlined terms in the list:

$$010001...$$

At a minimum, this sequence disagrees with the first sequence in the first term, disagrees with the second sequence in the second term, disagrees with the third sequence in the third term, and so on. So it's nowhere on the list. The same argument works for any list, which proves that the set of all binary sequences is uncountably infinite. Cantor's proof that $\mathbb{R}$ is uncountably infinite is similar.

So where does that leave us? We have divided the concept "infinite set" into "countably infinite set" and "uncountably infinite set", according to whether or not the elements can be "listed". The concept "uncountably infinite set" can be further subdivided (into infinitely many subdivisions, in fact), but to see that you'll have to take SM291!

World's Largest Prime Number Discovered

The Greek mathematician Euclid, who lived around 300 B.C., proved that there are infinitely many prime numbers. That doesn't mean they are easy to find, and every time a new largest prime is discovered, it's newsworthy (at least to mathematicians). Recently, a new largest prime was found on a UCLA Math Department computer. It was a *Mersenne prime*, a prime of the form $2^n - 1$, where n is an integer. For example, $3 = 2^2 - 1$, $7 = 2^3 - 1$, and $31 = 2^5 - 1$ are all Mersenne primes. It turns out that for $2^n - 1$ to have a shot at being prime, n itself must be prime. But just because n is prime, it doesn't mean that $2^n - 1$ will be prime. For example, $2^{11} - 1 = 2,047 = 23 \times 89$ is composite, even though 11 is prime. In fact, there are only 46 known Mersenne primes, and it is unknown whether there are infinitely many Mersenne primes! So what is the largest-known prime? As of August 2008, it is

$$2^{43,112,609} - 1,$$

a number with about 12.9 million digits! It was discovered using software which is freely available from `http://www.mersenne.org`, homepage of the Great Internet Mersenne Prime Search (GIMPS), brainchild of George Woltman. Join the search and you could gain both fame and fortune (sort of).

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