# RAYLEIGH PROBABILITY DISTRIBUTION APPLIED TO RANDOM WAVE HEIGHTS

In 1952, a British mathematician/oceanographer Prof. M.S. Longuet-Higgins, showed that random wave heights, H, followed the Rayleigh Probability Distribution (named for Lord Rayleigh who showed its applicability to the amplitude of sound waves in 1877).

**Rayleigh Probability Density Function**

The distribution of random wave heights may be described by a Rayleigh pdf with any of the following forms:

$$f(H) = \frac{H}{H_{\mathrm{mode}}^2} \exp\left(-\frac{H^2}{2H_{\mathrm{mode}}^2}\right)$$

$$f(H) = \frac{\pi}{2}\frac{H}{H_{mean}^2} \exp\left(-\frac{\pi}{4}\frac{H^2}{H_{mean}^2}\right)$$

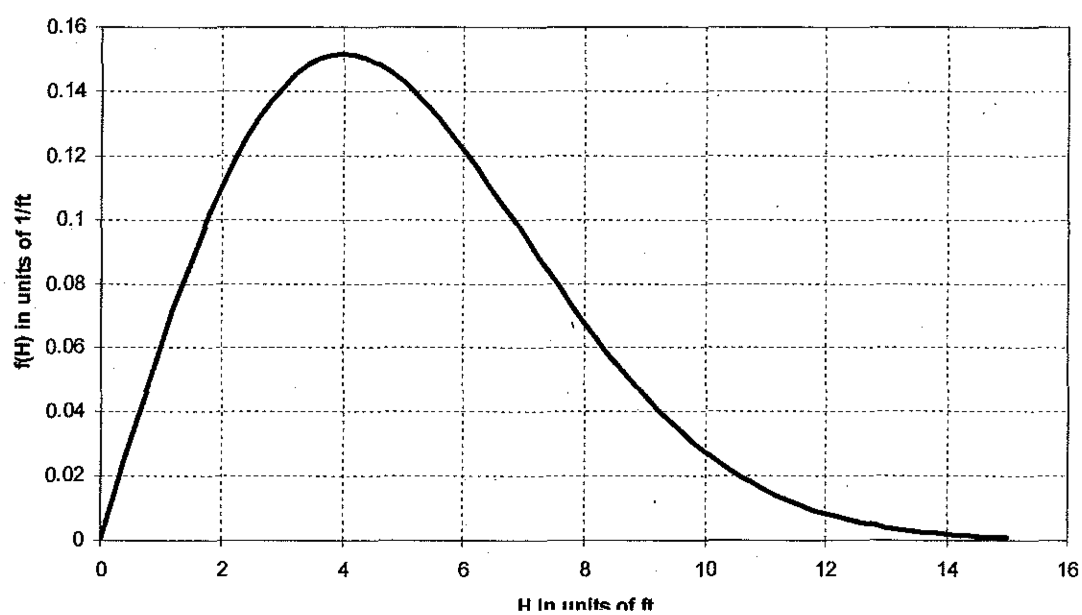
$$f(H) = 2\,\frac{H}{H_{RMS}^2} \exp\left(-\frac{H^2}{H_{RMS}^2}\right)$$

where the random values of H can be found once one of the following basic statistical measures is known:

$H_{mode}$ = modal or most common wave height
$H_{mean}$ = mean or average wave height
$H_{RMS}$ = root-mean-square wave height.

Basic properties of the Rayleigh distribution indicate the following (given here without proof):

$$H_{RMS} = \frac{2}{\sqrt{\pi}} H_{mean} = \sqrt{2}\ H_{\mathrm{mode}}$$



**Rayleigh Probability of Exceedence**

The area under f(H) to the right of any value of H gives the probability of waves being greater than H. This is often called the probability of exceedance, Q(H), and can be computed as:

$$Q(H) = \int_H^{\infty} f(H)\, dH$$

This can be found using any of the three basic parameters $H_{mode}$, $H_{mean}$ , or $H_{RMS}$. The simplest form uses $H_{RMS}$ and is given by

$$Q(H) = \exp\left\{-\frac{H^2}{H_{rms}^2}\right\}$$

Note that $H_{rms}$ is treated as a constant in the above equation and is used as a scaling parameter to describe the general size of waves in the sea state. The random variable is wave height, *H*, and the equation the gives the probability of wave heights being equal to or greater than any value *H*.

Note that the probability of waves exceeding $H_{RMS}$ is then $Q(H_{RMS}) = \exp(-1) = 0.368$.

Or, in a sea with, say $H_{rms} = 2.66$ meters, the probability of heights exceeding 5 meters is $Q(5m) = \exp(-(5.0/2.66)^2) = 0.029$ or 2.9 percent.

**Relationship Among Wave Height Statistics**

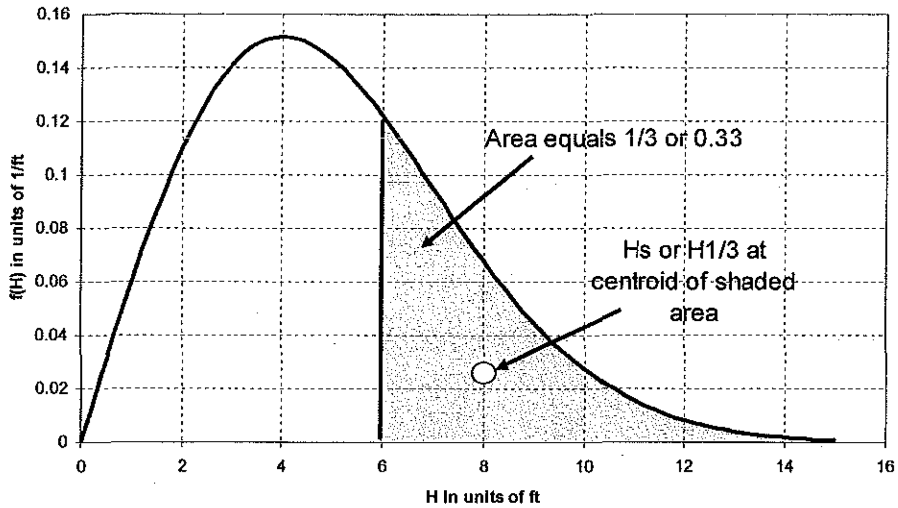
In addition to $H_{RMS}$, oceanographers, ocean engineers, and naval architects routinely use a different statistical wave height – the significant wave height, $H_S$ or $H_{1/3}$ - to describe the general height of waves in a random sea.

The significant wave height is defined as the average height of the largest 1/3 of the waves in the random sea, or of the top 33% of the waves in the random sea.

The significant height can be derived from the Rayleigh pdf. Referring to the next figure, Hs can be found by recognizing that: (1) we want to consider the portion of the pdf representing the top 1/3 or 33% of waves which is shaded in the figure and (2) the significant height is the <u>average</u> of these waves so it is the value of H at the centroid of the shaded area.

Working out the details, it can be shown that:

$$H_S = \sqrt{2}\, H_{RMS} = 1.414\, H_{RMS}$$




Similar arguments can be applied to find other wave statistics. For example, the average of the top 10% or 1/10 of the waves is found as the centroid of the top 10% of the area under the Rayleigh pdf. The result is:

$$H_{1/10} = 1.27 H_S = 1.80\ H_{RMS}$$

The average of the top 1% or 1/100 of the waves is found as the centroid of the top 1% of the area under the Rayleigh pdf as

$$H_{1/100} = 1.67 H_S = 2.36\ H_{RMS}$$

In addition, it has been shown that a good estimate of the maximum wave height in a wave record may be found from

$$H_{\max} = \sqrt{\ln J}\ H_{rms} = 0.707\sqrt{\ln J}\ H_S$$

where J is the number of waves in the random sea. For example, the maximum wave height in a random sea that had J=2000 waves and a significant wave height, Hs = 10m, would be estimated as

$$H_{\max} = 0.707\sqrt{\ln 2000}\ (10m) = 19.5m$$

As a Rule-of-Thumb, the maximum wave height is usually about two-times the significant wave height, i..e. $H_{max} \sim 2\ Hs$.

It is noted, however, that larger and more rare or extreme waves can occur. Rogue or freak waves are sometimes defined as individual waves whose height is larger than 2Hs or 2.5 Hs. From the Rayleigh distribution, the probability of a wave exceeding 2.5 Hs is given as

$$Q(2.5H_S) = \exp\left\{-\frac{(2.5H_S)^2}{H_{rms}^2}\right\} = \exp\{-12.5\} = 0.00000374$$

Thus, taking the inverse, we would expect only one wave out of 267,326 waves to exceed 2.5 Hs.