Chapter 5

The Output: Ship Motions in Waves

Learning Objectives:

1. Calculate the ship motion response spectrum given a sea state and relevant ship transfer function.
2. Calculate the motion variance from the motion response spectrum
3. Describe the difference between a transfer function and a Response Amplitude Operator (RAO)
4. Calculate the significant motion amplitude given the motion response spectrum
5. Calculate the probability of a particular motion amplitude being exceeded given the motion response spectrum
6. Identify the worst operating conditions given a polar plot with motion RMS or significant amplitudes
7. Explain what a Safe Operating Envelope is and how it is calculated.
8. **Laboratory Objectives:**
   (a) Calculate the significant pitch and vertical acceleration motions from an experimental time history record
   (b) Compare analytical, experimental, and simulation predictions for ship motions in irregular seas
   (c) Develop a realistic test plan for a seakeeping experiment

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Now that we have considered the input (waves) and the system characteristics (ship transfer functions), we are ready to put them together and determine ship motions in a realistic seaway! If we were to know the exact waves the ship will be encountering and we have access to a sophisticated CFD (computational fluid dynamics) code designed for seakeeping, we could predict (with reasonable accuracy) the actual motions the ship will make. However, we rarely know what exact waves the ship will encounter. Remember from Chapter 3 that when we deal with realistic sea conditions we have to work with probabilistic predictions. So, the best we will be able to do in terms of ship motions is predict the probable resulting motions for a given ship in a given wave spectrum.

5.1 The Electronic Filter Analogy

In 1951, St. Denis and Pierson suggested that the ship could be treated in much the same way as the "black box" in an electrical filter.

$$\text{Input} \rightarrow \boxed{\text{Filter}} \rightarrow \text{Output}$$

$$\text{Waves} \rightarrow \boxed{\text{Ship}} \rightarrow \text{Ship Motion}$$

The ship is a "black box" that receives the waves as input and generates ship motions as output. This analogy works as long as the filter is "linear." In other words, the output amplitude must be proportional to the input amplitude.

Consider the heave transfer function for a ship in head seas (see Figure 5.1). This transfer function looks much the same as a "low-pass" filter. For low ω_e the waves are translated into corresponding motions (same amplitudes and phase). For high ω_e, there are no resulting ship motions (i.e. the signal does not pass through the filter).

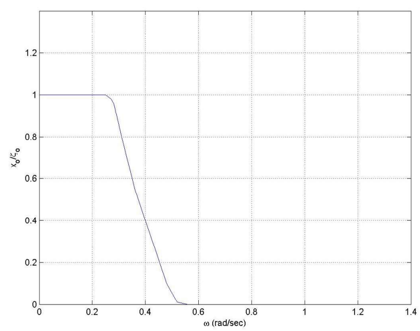


Figure 5.1: Typical Heave Transfer Function

5.2 First Challenge: Encounter Frequency Spectrum

The first challenge to apply the electronic filter analogy is that the ship experiences the **encounter frequency spectrum**, not the wave energy spectrum! Assuming the waves are long crested, the wave energy spectra formulae give the wave energy spectum for a fixed point in the ocean. We are going to need to transform this information to the reference frame of the moving ship. Remember the encounter frequency is found from

$$\omega_e = \omega - \frac{\omega^2 U \cos\mu}{g}$$

where encounter frequency (ω_e) is greater than the wave frequency (ω) in head seas and (generally) less than the wave frequency in following seas. Therefore, the wave energy spectrum will be *shifted* along the frequency axis to a *different* range of frequencies. For example, a typical open ocean wave spectrum might look like Figure 5.2. If we consider a ship operating in head seas for this wave energy spectrum, we would get an encounter frequency spectrum like the one shown in Figure 5.3. When we shift the wave frequency range from the wave energy spectrum to the equivalent encounter frequencies, we get the shaded areas in Figure 5.4. The relationship between the frequency spacing on the wave energy spectrum and the encounter frequency wave spectrum is given by

$$\delta\omega_e = \left(1 - \frac{2\omega U}{g}\cos\mu\right)\delta\omega.$$

The energy in a given range of wave frequencies must match the energy in the shifted set

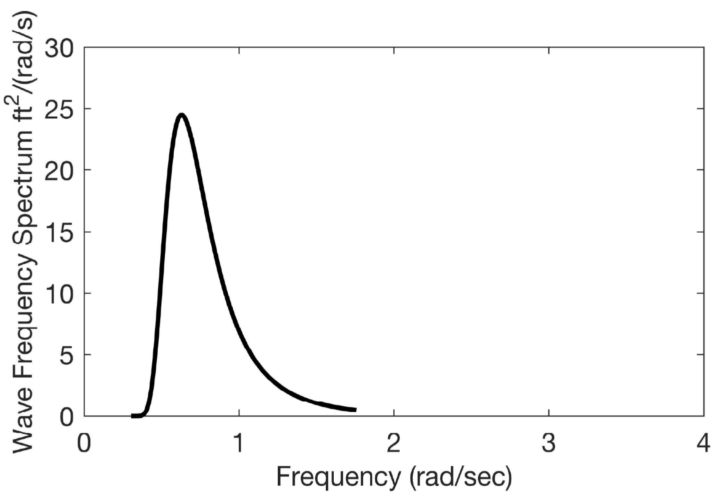


Figure 5.2: Typical Wave Energy Spectrum

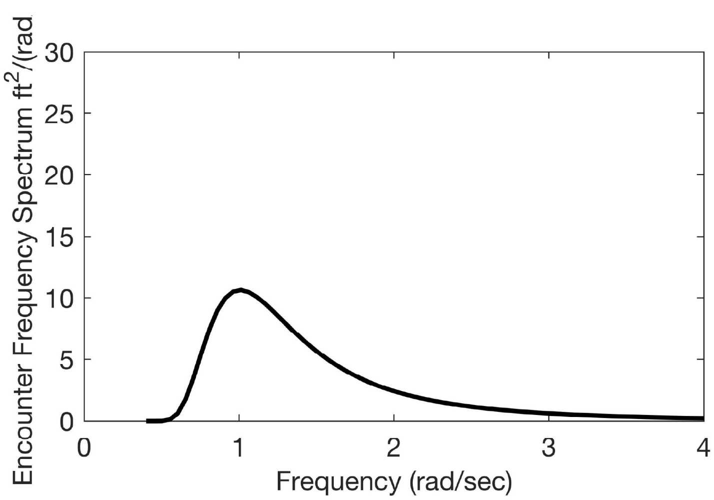


Figure 5.3: Encounter Frequency Spectrum

of encounter frequencies. In other words, the area in the shaded regions of Figure 5.4 must be equal. The areas must be the same since the total wave energy and the significant wave height must be the same whether the waves are measured by a stationary probe or one moving with the ship. So, the area of a section is,

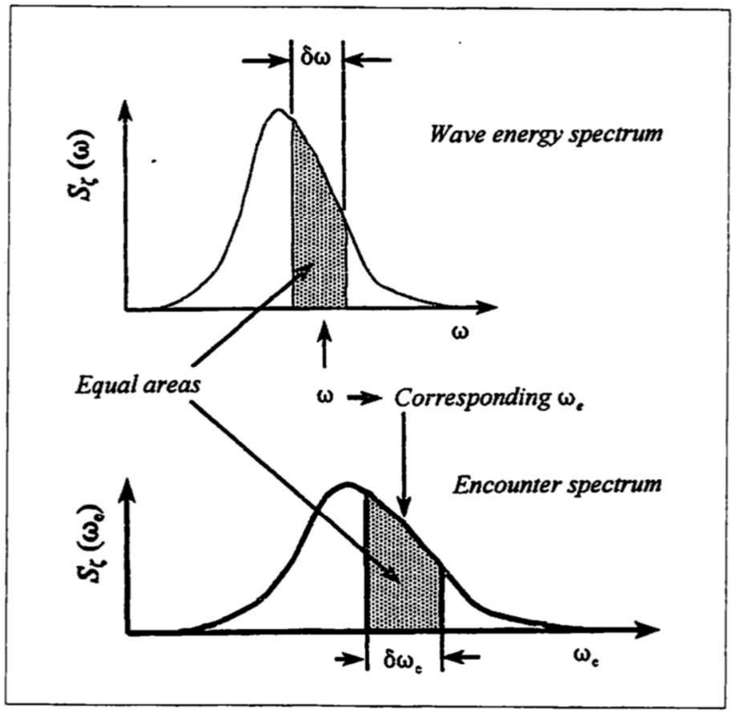


Figure 5.4: Areas under Wave Energy and Encounter Frequency Spectra (Figure 8.2 in reference 2)

$$S_\zeta(\omega_e)\delta\omega_e = S_\zeta(\omega)\delta\omega$$
$$S_\zeta(\omega_e) = S_\zeta(\omega)\frac{\delta\omega}{\delta\omega_e}$$

Plugging in the expression for $\delta\omega_e$ given above, the encounter wave energy spectrum can be found from

$$S_\zeta(\omega_e) = S_\zeta(\omega)\frac{g}{g - 2\omega U \cos\mu} \tag{5.1}$$

In head seas, the frequencies increase and the range widens which also results in reducing the height of the spectral ordinates (to keep the area under the curve the same).

5.3 Second Challenge: Motion Energy Spectrum

The second (and main) challenge is to get from the encounter frequency wave spectrum to the ship motion energy spectrum. The electronic filter analogy means the encounter frequency wave energy spectrum is the input, the ship transfer function is the filter, and the ship motion energy spectrum will be our output. But how do we combine the input and the filter to arrive at the output? The equation is actually fairly simple, you multiply the encounter frequency wave energy value (at each encounter frequency) times the square of the ship transfer function value (at the same encounter frequency). For example, the heave motion energy spectrum is found from

$$S_{x_3}(\omega_e) = S_\zeta(\omega_e)\big(\frac{X_{30}}{\zeta_0}|_{\omega_e}\big)^2 \tag{5.2}$$

The square of the transfer function is also referred to as the Response Amplitude Operator (RAO). However, the RAO is sometimes used to refer to the transfer function directly (for example, as in Maxsurf Motions), so it is important to double-check when something is called an RAO which the term is actually referring to. To summarize, the ship heave motion energy spectrum is found from

$$S_{x_3}(\omega_e) = S_\zeta(\omega_e) RAO_{x_3}.$$

To find the ship pitch or roll motion energy spectrum we need to find the encounter *wave slope* energy spectrum. The wave slope energy spectrum is

$$S_\alpha(\omega) = \frac{\omega^4}{g^2} S_\zeta(\omega)$$

and the encounter wave slope energy spectrum is equal to

$$S_\alpha(\omega_e) = S_\alpha(\omega) \frac{g}{g - 2\omega U \cos\mu}.$$

The resulting heave, pitch, and roll motions will be sinusoidal (just as the irregular waves that are in the wave energy spectrum), so we can determine the variance (measure of data spread) for these motions,

Heave	$m_0 = \int_0^\infty s_{x_3}(\omega_e) \mathrm{d}\omega_e$
Pitch	$m_0 = \int_0^\infty s_{x_5}(\omega_e) \mathrm{d}\omega_e$
Roll	$m_0 = \int_0^\infty s_{x_4}(\omega_e) \mathrm{d}\omega_e$

With the variance (or zeroth spectral moment), we can use the same techniques as for ocean waves (see section 3.2.2) to determine information about the velocity, acceleration, and motions using spectral moments! Remember, the spectral moments of an energy spectrum are equal to

$$m_n = \int_0^\infty \omega_e^n S_{x_3}(\omega_e) \mathrm{d}\omega_e$$

The root-mean-square (RMS) motion values are equal to the square-root of the variances. So, the RMS value for the motion amplitude is $\sigma_0 = \sqrt{m_0}$, the RMS value for velocity is

$\sigma_2 = \sqrt{m_2}$, and the RMS value for the acceleration is $\sigma_4 = \sqrt{m_4}$. The mean motion period, mean peak motion period, and mean zero-crossing period have the same equations as the irregular waves in Chapter 3,

$$\bar{T} = 2\pi \frac{m_0}{m_1}$$
$$\bar{T}_p = 2\pi \sqrt{\frac{m_2}{m_4}}$$
$$\bar{T}_z = 2\pi \sqrt{\frac{m_0}{m_2}}$$

Figure 5.5 shows an example of a heave motion energy spectrum determined from an encounter wave energy frequency for a ship in head seas. When the energy in the encountered

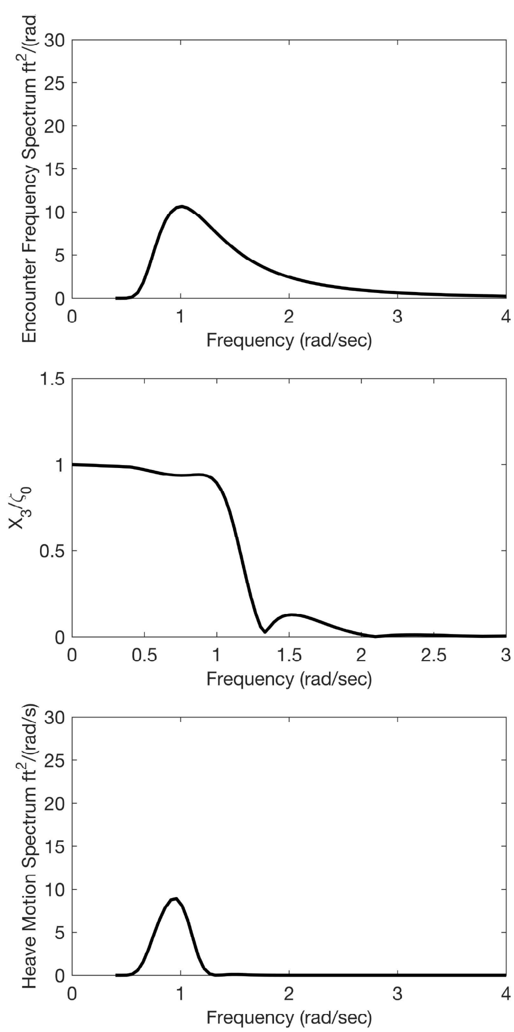


Figure 5.5: Calculation of heave energy spectrum

wave matches the largest response frequencies in the transfer function, you get large ship motions. When the energies do not overlap, only small ship motions result. Figure 5.6 shows the same ship transfer function (head seas at a single speed) when paired with different encounter frequency energy spectra (i.e. same ship operating in different sea conditions). As you can see, the ship response varies greatly based on what frequencies it is encountering in the seas. If you have large motions, one option is to change the encountered wave spectrum (change heading and/or speed) so that the high energy wave frequencies no longer align with the peak in your ship's transfer function.

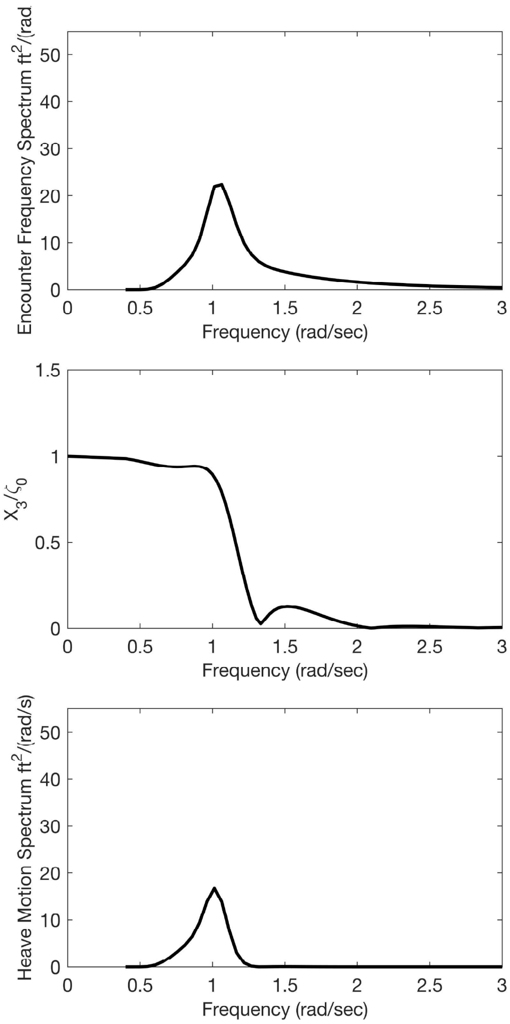


Figure 5.6: Heave motion energy spectra generated for the same ship transfer function but different encounter frequency energy spectra

5.4 Ship Motion Probability Distributions

Just as for the irregular waves, a key question in seakeeping is "what is the likelihood of a particular event occurring (such as a particular motion amplitude being exceeded)? We can use the ship motion energy spectrum to help us predict this type of probability.

Let's assume we are interested in the probability of the heave motion exceeding 2 m in a sea state 3. We can assume we have the relevant heave transfer function.

1. Begin by building an ITTC (Bretschneider) wave energy spectrum for a sea state 3 (determine the desired modal period and significant wave height from the NOAA tables). The NOAA tables show that the mean significant wave height for a SS3 is 2.85 ft and the most probable (modal) period is 7.5 sec. The equation we need is

$$S_\zeta(\omega) = \frac{1.25}{4}\left(\frac{\omega_0}{\omega}\right)^4 \frac{H_{1/3}^2}{\omega} e^{-1.25(\frac{\omega_0}{\omega})^4}$$

   where $\omega_0 = 2\pi/T_0$

2. Next convert the wave energy spectrum into the encounter frequency spectrum

$$S_\zeta(\omega_e) = S_\zeta(\omega)\frac{g}{g - 2\omega U\cos\mu}$$

3. Determine the RAO from the transfer function

$$RAO = \left(\frac{X_3}{\zeta_0}\right)^2$$

4. Find the motion energy spectrum

$$S_{x_3}(\omega_e) = s_\zeta(\omega_e) \cdot RAO_{x_3}$$

5. Find the variance (m_0) and RMS (σ_0)

$$m_0 = \int_0^\infty s_{x_3}(\omega_e)\mathrm{d}\omega_e$$
$$\sigma_0 = \sqrt{m_0}$$

6. Find the probability $X_3 > 2$ m

$$P(X_3 > 2\text{ m}) = e^{-\frac{1}{2}\left(\frac{x_3}{\sigma_0}\right)^2} = e^{-\frac{1}{2}\left(\frac{2}{\sigma_0}\right)^2}$$

Calculation of Probability of Exceedance Example Consider the pitching motion of a 520 ft ship. For the time history recorded (a total of 23.5 minutes), the variance of the pitching motion was 6.522 deg^2, the variance of the pitching velocity was 3.562 deg^2/sec^2, and the variance of the pitching acceleration was 2.276 deg^2/sec^4. Find the significant pitching magnitude and the probability of the pitching motion exceeding 6 degrees.
How did we get m_0? It was found either by taking the variance of the time history or by finding the area under the pitch motion energy spectrum. Once we have it we can find the RMS pitching motion from

$$X_{\text{RMS}} = \sqrt{m_0} = \sqrt{6.522\text{ deg}^2} = 2.55\text{ deg.}$$

The significant pitching amplitude is twice the RMS value, so

$$\bar{X}_{1/3} = 2(2.55\text{ deg}) = 5.10^\circ$$

To find the probability of exceedance we plug this into the equation 3.12 from Chapter 3.

$$P(X_5 > 6^\circ) = e^{-\frac{1}{2}(\frac{6^\circ}{\sigma_0})^2} = e^{-\frac{1}{2}(\frac{6^\circ}{2.55^\circ})^2} = 0.0628$$

So, for this sea condition, there is a 6.28% probability the pitch motion will exceed 6°.

5.5 Polar Plots

The Response Motion Spectrum is one way to describe the ship motion in a realistic sea state, but it is not easy to get useful information directly from the plot. To find information about the magnitudes of probable motions the area under the plot needs to be taken and that only gives information on a single wave/ship heading combination. To determine the ship response in other headings (or the same heading but at different speeds), the process needs to be repeated. To understand the ship response for all headings and multiple speeds you will need to find the information from many response spectrum plots. One way to consolidate the information is using a polar plot.

There are two main types of polar plots and both have the heading angle as the angle around the plot. In the version we will be using in class, the radius of the polar plot represents the speed of the ship and the color of the lines represent the magnitude of the response. (In the other type of polar plot the radius represent the response magnitude and each curve is for a different ship speed). Figure 5.7 shows a typical example of a polar plot. The magnitudes for this plot are the RMS responses. To find the significant amplitudes you need to double the values found on this plot.

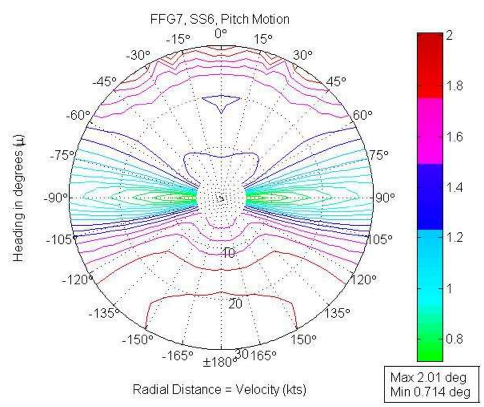


Figure 5.7: Sample Polar Plot

The information on the polar plot can be converted into a probability of exceedence based on the target magnitude plotted. For example, a plot of the significant motion amplitude gives a plot showing the motion amplitudes that have about a 14% chance of being exceeded. If desired, a polar plot showing the 10% or 1% chance of being exceeded could be displayed. The polar plot also has the advantage of showing under what operating conditions (speed and heading) for a particular sea the ship will have the largest motions. This will be true at any probability level. So, if you are experiencing large rolls in a quartering sea and your ship's polar plot for that condition shows beam seas are the worst likely response, it gives you better guidance on the best course of action for minimizing your ship's motion response.

5.5.1 Safe Operating Envelopes (SOE)

The calculations necessary for a polar plot are done using computational tools to determine response amplitudes for a large number of sea states, sea directions, and ship speeds. The results are often categorized by some limits (set by experience or come other guidance) and summarized in plots known as Safe Operating Envelopes (SOE). These polar plots show regions of marginal and unacceptable motions. Below are some examples from Comstock and Keane (1980) showing slamming predictions for CV-41 based on a change in sponson design[1].

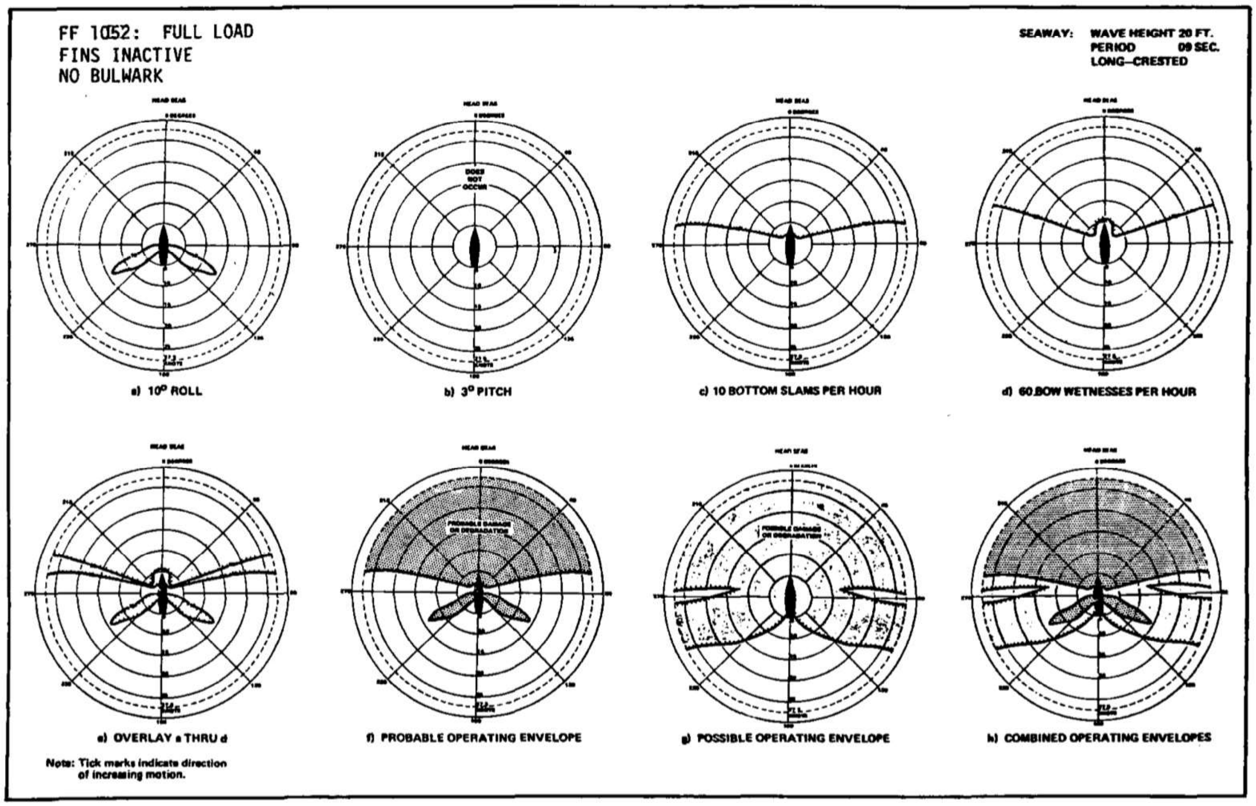


Figure 5.8: Development of FF 1042 Class Heavy Weather Seakeeping Operating Envelopes (SOEs)

5.6 Model Testing in Irregular Seas

Model testing in irregular head seas is a realistic scenario for seakeeping. The goal of seakeeping is to evaluate motions and accelerations in different sea conditions. Model testing is only possible for limited geometries and sea conditions, however, so it is important to be able to use other methods to predict motions and accelerations.

[1]Comstock, Edward N. and Keane, Robert G., Jr. (1980) "Seakeeping by Design," *Naval Engineers Journal*, vol. 92, no. 2, pp. 157-178.

Problems

1. Using the wave spectrum and heave transfer function given below,

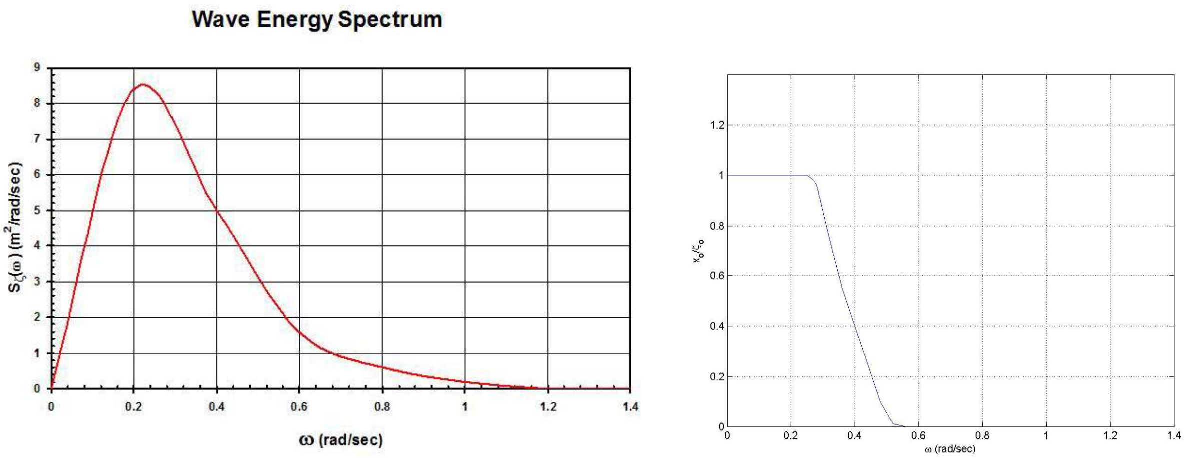


Figure 5.9: Wave Energy Spectrum (left) and Heave Transfer Function (right)

   (a) Determine the variance for the heave **motion** energy spectrum

   (b) Determine the RMS for the heave motion energy spectrum

   (c) Determine the RMS for the heave **acceleration** energy spectrum

2. Consider the wave energy spectrum and pitch transfer function provided below (beam seas, $\mu = 90°$) and find the RMS pitch motion.

Frequency (rad/sec)	$S_\alpha(\omega_e)$ rad^2/(rad/sec)	$x_{50}/(k\zeta_0)$
0	0	0
0.2	0	8.433
0.4	0.0001	2.11
0.6	0.0011	1.56
0.8	0.0047	0.88
1.0	0.0077	0.34
1.2	0.0120	0.23
1.4	0.0185	0.17
1.6	0	0

3. Consider the following information:

ω_e (rad/sec)	$S_\zeta(\omega_e)$ m^2/(rad/sec)	RAO_{x_3}
0.4	1	0
0.6	13	0.4
0.8	27	1.7
1.0	21.5	5.1
1.2	15	8.3
1.4	10	6.0
1.6	5.5	2.6
1.8	3.5	1.8
2.0	2.5	1.4

   (a) Find the RMS heave motion.

   (b) What is the probability the heave motion will exceed 14 m?

4. What is the difference between a transfer function and a RAO?

5. If the roll motion spectrum has an area of 0.1120 rad^2, what is the significant roll amplitude (in degrees)?

6. Consider the heaving motion of a 500 ft ship traveling at 15 knots at a heading of $\mu = 160°$. For a sea state with a significant wave height of 5 m and a modal period of 10.8 sec, the variance of the heave motion was 1.33 m^2, the variance of the heave velocity was 0.581 m^2/sec^2, and the variance of the heave acceleration was 0.293 m^2/sec^4. Find the significant heaving magnitude and the probability of the heaving motion exceeding 2.8 m.

7. The goal for this problem is to take a prescribed sea spectrum and calculate the resulting significant heave amplitude.

   (a) Calculate the heave motion spectrum for the DDG-51 in a Bretschneider spectrum with $H_{1/3} = 3.3$ ft and $T_0 = 5.6$ seconds. The spreadsheet on the course website (*ShipMotionSpectraAssignment.xlsx*) contains the relevant transfer functions. Complete this for both 15 knots and 20 knots forward speeds in head seas.

   (b) Plot the heave motion spectrum for each condition versus the encounter frequency.

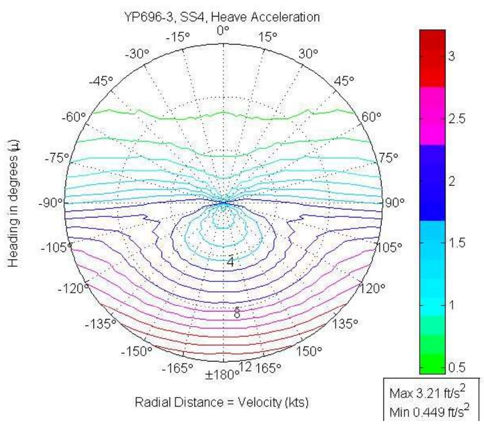
   (c) Check the significant wave height for the encounter spectrum by finding the area under the curve. Be sure it equals $H_{1/3} = 3.3$ ft.

   (d) Calculate the significant heave amplitude for the two conditions.

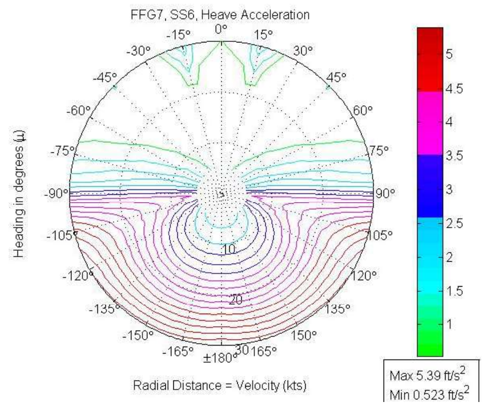
   (e) Calculate the probability the heave amplitude exceeds 1 ft.

8. Starting with a Wave Spectrum and the vessel response in regular waves as measured in a series of experiments (i.e. the transfer function), describe the procedure for finding the Motion Response Spectrum for the motion of interest.

9. Consider the polar plots provided below. What is the maximum RMS heave acceleration in g's? What does this mean the maximum significant heave acceleration is (in g's)? What is the average of the $1/10^{\text{th}}$ highest heave accelerations? Under what operating conditions does this maximum response occur? What operational guidance would you give for this vessel in this sea state?



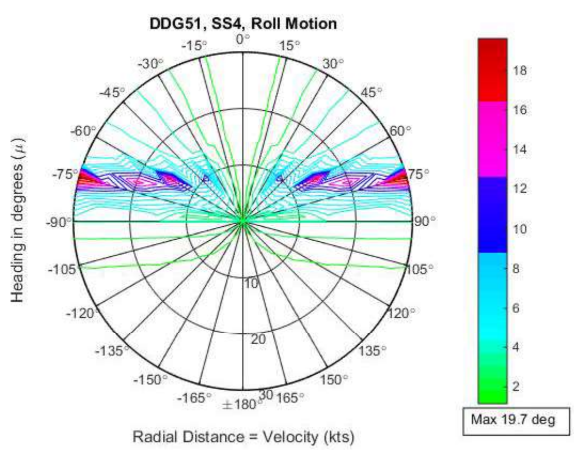
(a)



(b)

10. A ship sails in a natural seaway of approximately $T_e = 6$ sec encounter period between ship and waves. The bridge of the ship is located near the bow. During one hour, the variance of the acceleration time history was measured as 0.109 g^2. For downward accelerations exceeding 1.5 g's severe injuries for the crew have to be expected. What is the probability of this happening if the ship continues sailing with the same speed in the same seaway?

11. What is the significant roll amplitudes for the RMS roll response shown below?



12. To move from the sea state (wave spectrum) to the motion of the hull (motion response spectrum), two pieces of information are needed. Which are the best two?
    (i) wave spectrum and transfer function
    (ii) encounter wave spectrum and fourier transform
    (iii) encounter wave spectrum and transfer function
    (iv) wave spectrum and fourier transform