* Kinematics is the relationships between the positions, velocities and accelerations of the links of a manipulator.

* The problem we’re eventually trying to solve is: we know where we want our robot is, what position are the motors in? This is Inverse Kinematics, and is important because we can use the same technique to solve: we know what position we *want* to be in, what position would the motors be in? Once we know that, we can compare where the motors are now to where the motors should be, and make them so.

* To understand that, we first need to understand the problem of: We know what position the motors are in, where is the arm? This is know as the forward kinematics problem.
  
  - We will focus on the positions.
  
  - The velocities and accelerations are based on the first and second derivatives of the positions, naturally, but we won’t discuss them in this class.

* Serial link manipulator is a series of links which connects the hand of the robot to the base.
  
  - Each link is connected to the next by an actuated joint (i.e. one that the robot can move).
  
  - The relationship between neighboring links can be described with (yet another) homogeneous transformation matrix, denoted $A$.
  
  - We use a series of $A$ matrices to describe the transform from the base of the robot to the hand of the manipulator- This is called the forward kinematic transform.

* This transform is part of a closed equation involving the position transform from the previous discussion on space:

\[
^{R}T_{H} = ^{R}T_{L_{1}}L_{1}^{T}L_{2}L_{2}^{T}L_{3} \cdots L_{n-1}^{T}L_{n}
\]

we’ll abbreviate each $L_{m-1}^{T}L_{m}$ as $A_{m}$.

* In order to calculate the hand position, we must:
  
  - Set up coordinate frames for each link in the hand,
Generate \( \mathbf{A} \) matrices for each transformation between the frames,

- Multiply the \( \mathbf{A} \) matrices to generate the right hand side of the above equation,
- Solve the equation for the various pieces we need: the position of the hand frame, as well as its orientation.

- The method of setting up coordinated frames is somewhat arbitrary, but we'll use one standard (Paul, 1981).

- **Setup:**
  - Define: **Link**- a rigid piece of the robot arm that connects two joints. It therefore maintains a fixed relationship between the joints at its ends.
  - Define: **Joint**- a connection between two links which allows the links to either rotate or translate w/r/t each other. When they rotate, it is called a revolute joint, when they translate, its called a prismatic- no idea why. We will mostly discuss revolute joints cause that’s what we have in our robots. But note that the fingers are prismatic.
  - Define: **Base**- a link that does not move w/r/t the robot frame.
  - Define: **Proximal**- of two things (links, joints etc.) the one closer to the robot base in the chain of links.
  - Define: **Distal** - of two things (links, joints etc.) the one farther from the robot base in the chain of links.
  - Define: **Joint Axis**- the axis around which the revolute joint turns.
  - Start at the base, number the links from 0 to n. The base is 0.
  - Number the joints from 1 to n+1. The endpoint of the robot arm is n+1.
  - Base frame will be located at joint 1. Why? Why not the bottom of the base?
  - z-axis of the base frame is the joint axis of joint 1.
  - x-axis of the base frame should be in the direction you want to make the start position of link 1.

- **Repeatedly assign coordinate frame to each link:**
  - To locate the origin of the frame:
    * identify the 2 joint axes of the joints at either end of the link.
    * If the joint axes do not intersect:
      · There should be a line which passes through the origin of the previous coordinate frame that is perpendicular to both joint axes.
      · That line is called the common normal to the two joint axes.
      · place the origin of the frame at the intersection of the common normal and the distal joint axis.
• if the joint axes do intersect, then locate the origin at the intersection of the joint axes.

  – Make the z axis the distal joint axis. What direction should the z be? It doesn’t really matter, but its best if we’re consistent as a class. We’ll start with z pointing up, and talk about other cases later.

  – If the 2 joint axes do not intersect, then make the x axis coincident with the the common normal to the 2 joint axes of the link. Make x point away from the proximal joint.

  – If the joint axes intersect, then make the x axis perpendicular to the plane defined by the 2 joint axes.

  – Make the y axis correct using the right hand rule.

  – At the hand/gripper set a coordinate frame at the center of the gripper area (the grip location)

  * make the z-axis parallel to the z-axis of the previous coordinate frame.
  * make the z-y plane parallel to the plane of the hand.
  * Set x-axis so that is is parallel and in the same direction as the previous x-axis.

• See some examples.
Look at the examples above and note that the relation between two adjacent coordinate frames can be described by the following parameters:
- $l_n$ - the length parameter. The distance between the two $z$ axes. Note that if the $z$ axes intersect, then $l_n = 0$.
- $d_n$ - the distance parameter. The distance between the two $x$ axes. Often this too can be 0.
- $\theta_n$ - the link angle. The angle between link $n-1$ and link $n$.
- $\alpha_n$ - the twist. The angle of twist between the two joints in the link itself.

- We define the $A$ matrices using these parameters:

\[
A_n = \text{Rot}(z, \theta) \text{Trans}(0,0,d) \text{Trans}(l,0,0) \text{Rot}(x, \alpha).
\]

\[
\begin{bmatrix}
\cos(\Theta) & -\sin(\Theta) & 0 & 0 \\
\sin(\Theta) & \cos(\Theta) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & l \\ 0 & 1 & 0 & 0 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) & 0 \\
0 & \sin(\alpha) & \cos(\alpha) & 0 \\
0 & 0 & 0 & 1 \end{bmatrix}
\]

- Finally, we multiply the $A$ matrices together to get the transformation $T$: $T = A_1A_2A_3...A_n$

- Next the question becomes, given a $T$ calculated by multiplying the $A$ matrices, can we determine the position and orientation of the end of the arm?

- given a $T$:

\[
\begin{bmatrix}
x_x & y_x & z_x & p_x \\
x_y & y_y & z_y & p_y \\
x_z & y_z & z_z & p_z \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

calculated by the product of $A$’s,

- recall that $T$ is also the product of translation and rotation from the reference frame:

$T = \text{Trans}(x,y,z)\text{Rot}(z,\phi)\text{Rot}(y,\beta)\text{Rot}(x,\psi)$.

which is:

\[
\begin{bmatrix}
\cos(\phi)\cos(\beta) & \cos(\phi)\sin(\beta)\sin(\psi) - \sin(\phi)\cos(\psi) & \cos(\phi)\sin(\beta)\cos(\psi) + \sin(\phi)\sin(\psi) & p_x \\
\sin(\phi)\cos(\beta) & \sin(\phi)\sin(\beta)\sin(\psi) + \cos(\phi)\cos(\psi) & \sin(\phi)\sin(\beta)\cos(\psi) - \cos(\phi)\sin(\psi) & p_y \\
- \sin(\beta) & \cos(\beta)\sin(\psi) & \cos(\beta)\cos(\psi) & p_z \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
We set our 2 versions of $T$ equal to each other and solve for the position $p$ as well as $\phi, \beta,$ and $\psi$.

$$
\beta = \arcsin(-xz) \\
\psi = \arcsin\left(\frac{yz}{\cos(\beta)}\right) \\
\phi = \arcsin\left(\frac{xz}{\cos(\beta)}\right)
$$

And $p$ is just the rightmost column.