Intelligent behavior can only be gained in robots if they are responding to inputs from their sensors. In other words, they have to have some understanding of the world in order to behave appropriately. Unfortunately, sensors are unreliable; things go wrong frequently, meaning that depending too much on any given reading is a recipe for failure.

So, if we can’t trust our sensors, what hope is there? Imagine using a rangefinder, which can tell distance from the robot to a wall. On five consecutive readings from a stationary robot, it tells us the wall is 105 cm, 92 cm, 506 cm, 103 cm, and 98 cm away. Our sensor is clearly jumping all over the place. But, is there truly nothing we can learn from these readings? Of course not. In fact, it is likely obvious to you that the wall is likely somewhere around 100 cm away. You might even say that it is probable that the wall is close to 100 cm away.

We can formalize this intuition by using the mathematical field of probability. Eventually we’ll understand this well enough to do some exciting things, like mapping, even with unreliable sensors.

**Basic Probability**

- $P(H \leftarrow a)$ is the probability of some variable H having value a.
- e.g. $P(\text{CoinFlip} \leftarrow \text{heads})$, $P(\text{CellOccupied} \leftarrow \text{true})$ This is called a hypothesis.
- The hypothesis has some number $p$, $0 \leq p \leq 1$: $P(\text{CoinFlip} \leftarrow \text{heads}) = 0.5$, $P(\text{CellOccupied} \leftarrow \text{true}) = 0.33$.
- The number is a prediction of the outcome of the event. These numbers reflect a measure of the likelihood that this outcome will be the outcome of the event. It’s a prediction of the future. If it is a repeating event, it says, “if this same situation comes up n times, then we expect this to be the outcome $p \times n$ times.” Thus if the probability of me having a good day is 0.6, then we would expect me to have 3 good days every 5 day work week.
- The sum of the $p$'s across all values of the variable must be 1.: $P(\text{CoinFlip} \leftarrow \text{heads}) = 0.5$ and $P(\text{CoinFlip} \leftarrow \text{tails}) = 0.5$ or $P(\text{CellOccupied} \leftarrow \text{true}) = 0.33 + P(\text{CellOccupied} \leftarrow \text{false}) = 0.67$.
- If the values are true or false, we often use a shorthand: $P(\text{CellOccupied} \leftarrow \text{true})$ becomes $P(\text{CellOccupied})$ and $P(\text{CellOccupied} \leftarrow \text{false})$ becomes $P(\neg \text{CellOccupied})$
- the set of $p$'s for all values that H can be is called the probability distribution.
- When we say $P(A)$ we mean not $P(A \leftarrow \text{true})$ but we mean the whole probability distribution: $P(A \leftarrow \text{true}) \land P(A \leftarrow \text{false})$ or $P(A \leftarrow 1) \land P(A \leftarrow 2) \land P(A \leftarrow 3)$.
- be careful of that.
- $P(A \land B) = P(A)P(B)$
- $P(A \lor B) = P(A) + P(B)$
- $P(A) + P(\neg A) = 1$. Some of what I said above applies only to cases where two variables are are independent.
  - Variables are independent when the value of one does not affect the value of another: $A \perp B$.
  - When flipping 2 different coins, the value of one does not affect the value of the other.
  - But let's take a different example:
    - * Imagine we live in a town where it rains 50% of the time.
    - * $P(\text{Raining}) = 0.5$ ($P(\text{Raining} \leftarrow \text{true}) = 0.5$)
* Most of the time it rains, the sidewalk gets wet. Sometimes it doesn’t because the tent for the local art fair keeps it dry.
* \( P(Wet) = 0.49 \)
* If we use the rule \( P(A \land B) = P(A)P(B) \), then the \( P(Raining \land Wet) = 0.245 \).
* That says that \( \frac{1}{4} \) of the time, it will be raining and the sidewalks will be wet, because there is a causal relationship between these two events.
* But does that make sense? You expect that \( \frac{1}{2} \) of the time it will be raining, and nearly 100% of the time it is raining, the sidewalks will be wet.
* It doesn’t make sense, so instead we use a rule based on Conditional Probability.

Conditional Probability is a way of expressing the relationship between two variables, expressed as: \( P(A \leftarrow x|B \leftarrow y) \) The probability of \( A \) being \( x \), given that \( B \) is \( y \), or more succinctly: \( P(A|B) \), the probability of \( A \) given \( B \).

Properties of conditional probability:
* \( P(A|B) = P(A) \) if \( A \perp B \). In general, we assume dependence unless we know independence.
* \( P(A \land B) = P(A)P(B|A) \) Note that the independent case of conjunction derives from these two: \( P(AB) = P(B)P(A|B) = P(B)P(A) \).
* \( P(Raining \land Wet) = P(Raining)P(Wet|Raining) \)
* \( P(Raining \land Wet) = 0.5 \times 0.96 = 0.48 \)
* for more than one variable:

\[
P(A \land B \land C) = P(ABC) \text{ Just notationally more compact.} \\
P(AB)P(C|AB) \\
P(C|AB)P(B|A)P(A)
\]

- If \( P(A \land B) = P(A)P(B|A) \), then \( P(B \land A) = P(B)P(A|B) \)
- \( P(A \land B) = P(B \land A) \)
- \( \therefore P(A)P(B|A) = P(B)P(A|B) \)
- If we solve for \( P(A|B) \) we get: \( P(A|B) = \frac{P(A)P(B|A)}{P(B)} \)
- If that is true, then \( P(\neg A|B) = \frac{P(\neg A)P(B|\neg A)}{P(B)} \) too. These are known as Bayes’ Rule.

**Inferring Knowledge about the World**

As an example, let’s start with a very simple example (scenario cribbed from Thrun et al. “Probabilistic Robotics”). We’ll learn some things about probability along the way.

Suppose we have a robot, capable of opening a door. It also has a sensor, which tells it if the door is open or closed. The notation we’ll use is as follows: \( o \) will denote our observation, and \( x \) will denote our “state”, meaning the condition of reality (in this case, our state is if the door is open or closed).

Now, this sensor is unreliable (or “noisy”). If the door is closed, it gets this wrong, and observes it as open 20% of the time. We can denote this as follows:

\[
p(o \leftarrow x | x \leftarrow \text{closed}) = 0.2 \\
p(o \leftarrow x | x \leftarrow \text{open}) = 0.8
\]

We read this as “the probability that the observation is closed, given that the door is closed, is 0.8.” This is known as a conditional probability, because the probability of the thing before the vertical line being true is conditioned upon the thing after the vertical line being true.

Let’s go back to our sensor. If the door is actually open, it gets this wrong and observes it as closed 40% of the time.

\[
p(o \leftarrow x | x \leftarrow \text{open}) = 0.6 \\
p(o \leftarrow x | x \leftarrow \text{closed}) = 0.4
\]
What we are interested in finding out is given our readings, what is the likelihood that the door is open? In other words, we want:

\[ p(x \leftarrow \text{open}|o). \]

Going from \( p(a|b) \) to \( p(b|a) \) is the application of Bayes Rule that we derived above. Let’s try this. Let’s say that at the first timestep, our robot’s sensor takes a reading, and observes the door is open (\( o \leftarrow \text{open} \)). What is the most likely state \( x \)?

We only have two possible states, so what we want to calculate is \( p(x \leftarrow \text{open}|o \leftarrow \text{open}) \) and \( p(x \leftarrow \text{closed}|o \leftarrow \text{open}) \).

Let’s apply Bayes’ rule.

\[
p(x \leftarrow \text{open}|o \leftarrow \text{open}) = \frac{p(o \leftarrow \text{open}|x \leftarrow \text{open})p(x \leftarrow \text{open})}{p(o \leftarrow \text{open})}
\]

\[
= \frac{0.6p(x \leftarrow \text{open})}{p(o \leftarrow \text{open})}
\]

\[
p(x \leftarrow \text{closed}|o \leftarrow \text{open}) = \frac{p(o \leftarrow \text{open}|x \leftarrow \text{closed})p(x \leftarrow \text{closed})}{p(o \leftarrow \text{open})}
\]

\[
= \frac{0.2p(x \leftarrow \text{open})}{p(o \leftarrow \text{open})}
\]

We still have two unknowns. The first is our priors \( p(x \leftarrow \text{open}) \) and \( p(x \leftarrow \text{closed}) \). Now, suppose we have some knowledge like that the door is closed 90% of the time; this is where this human knowledge could come into play, by making \( p(x \leftarrow \text{open}) = 0.1 \). This would skew the results such that one unreliable sensor reading wouldn’t overcome our prior belief that the door is probably closed. In this case, we’ll assume that our belief is that the door is open about half the time.

\[
p(x \leftarrow \text{open}|o \leftarrow \text{open}) = \frac{0.6 \times 0.5}{p(o \leftarrow \text{open})}
\]

\[
p(x \leftarrow \text{closed}|o \leftarrow \text{open}) = \frac{0.2 \times 0.5}{p(o \leftarrow \text{open})}
\]

As always, “there’s an XKCD for that,” even for the use of a prior in statistics. See Figure 1. People who do statistics with priors using Bayes’ rule are known as “Bayesians,” while those who don’t are called “frequentists.” The holy war between the two rivals that of the operating systems war, or even the vi vs. Emacs war.

OK, so we’re nearly there. Now, we just have to fill in \( p(o \leftarrow \text{open}) \), and we’ll have calculated our posteriors. Sometimes we’ll know that number, and sometimes we won’t. Fortunately, we never have to care. Because the door MUST be either open or closed, \( p(x \leftarrow \text{open}|o) + p(x \leftarrow \text{closed}|o) = 1 \). See why? So, we can say that

\[
\frac{0.6 \times 0.5}{p(o \leftarrow \text{open})} + \frac{0.2 \times 0.5}{p(o \leftarrow \text{open})} = 1
\]

Work the math through, and we get that \( p(o \leftarrow \text{open}) = .4 \) (try to avoid gaining any intuition from this number, we only care about the conditional probability of the door being open). So,

\[
p(x \leftarrow \text{open}|o \leftarrow \text{open}) = \frac{0.6 \times 0.5}{0.4}
\]

\[
= .75
\]

\[
p(x \leftarrow \text{closed}|o \leftarrow \text{open}) = \frac{0.2 \times 0.5}{0.4}
\]

\[
= .25
\]

So, we can conclude there is a 75% chance the door is actually open, given our sensor reading.

Now, we’ll have the robot take the action of opening the door. We’ll denote the state before the action as \( x_{t-1} \) and the state after as \( x_t \) (because they’re the states at timesteps \( t - 1 \) and \( t \), respectively). We’ll
notate the action with $u$. Like our sensor, our actuator is “noisy,” meaning it isn’t always effective. In fact, we have the following probabilities:

\[
p(x_t \leftarrow \text{open}|u \leftarrow \text{push}, x_{t-1} \leftarrow \text{open}) = 1
\]
\[
p(x_t \leftarrow \text{closed}|u \leftarrow \text{push}, x_{t-1} \leftarrow \text{open}) = 0
\]
\[
p(x_t \leftarrow \text{open}|u \leftarrow \text{push}, x_{t-1} \leftarrow \text{closed}) = 0.8
\]
\[
p(x_t \leftarrow \text{closed}|u \leftarrow \text{push}, x_{t-1} \leftarrow \text{closed}) = 0.2
\]

Again, we gather these numbers experimentally.

So, we took a sensor reading, determined there was a 75% chance of it being open, and now we’re going to push on the door. What is the probability the door is open? That is, what is $P(x_t|u_{t-1} \leftarrow \text{push})$?

For this, we’ll use the following fact:

\[
p(a) = \sum_b p(a|b)p(b).
\]

So,

\[
p(x_t \leftarrow \text{open}|u \leftarrow \text{push}) = p(x_t \leftarrow \text{open}|u \leftarrow \text{push}, x_{t-1} \leftarrow \text{open})p(x_{t-1} \leftarrow \text{open})
\]
\[
+ p(x_t \leftarrow \text{open}|u \leftarrow \text{push}, x_{t-1} \leftarrow \text{closed})p(x_{t-1} \leftarrow \text{closed})
\]
\[
= 1 \times 0.75 + 0.8 \times 0.25
\]
\[
= 0.95
\]
So, there is now a 95% chance that the door is open. This can now be used as the prior for the next application of Bayes rule.

This algorithm will guide us for a while. We start with some belief about the world, we take an action, which changes our belief, we take a sensor reading, which changes our belief again. We then repeat. This is sometimes known as a **Bayes Filter**. They call it a “filter” because it filters out the noise in our sensor readings.

Those of you who took Artificial Intelligence will recognize what we just did as the forward algorithm for a Hidden Markov Model (HMM). That algorithm is the generalization of the example we just developed. In general, the world is modeled as the following diagram:

The true state of the world evolves over time, represented by the $X$ variables, but we only see what our sensors tell us, represented by the $e$ evidence variables. The general form of the algorithm then is:

$$P(X_t|e_{1:t}) = \alpha P(e_t|X_t) \sum_{X_{t-1}} P(X_t|X_{t-1}, U_{t-1})P(X_{t-1}|e_{1:t-1})$$