The Solow model (described in Weil Chapter 3) and the Romer model (described in Weil Chapter 8) can be combined in a relatively straight-forward way. In this model, both capital accumulation and technological growth are “endogenized.”

The model can be summarized by just three equations.

1) \[ Y = AK^\alpha [(1 - \gamma_A) L]^{1-\alpha} \]

2) \[ \Delta k = \gamma_Y y - \delta k \]

3) \[ \hat{A} = \left( \frac{1}{\mu} \right) \gamma_A L \]

The first equation describes how GDP is produced – technology combines with capital and production workers. But now two things in this economy grow – capital and technology.

The second equation describes the growth of capital – this is the law of motion for capital we discussed in Chapter 3, where capital rises due to savings (the savings rate is now called \( \gamma_Y \)) and falls due to capital depreciation.

The third equation describes the growth of technology – this is the equation we discussed in Chapter 8, where technological growth rises with the number of “researchers” and falls with the “cost of invention” (\( \mu \)).

Notice that now there are two forms of “savings” - \( \gamma_Y \), which is the fraction of output per person that is not consumed, and \( \gamma_A \), which is the fraction of the population that does not work in production but instead works in research.

Long-run Growth

We want to ask how fast living standards rise in this model (that is, what is \( \hat{y} \)?) First, rewrite equation 1) in per person terms:

4) \[ \hat{y} = A \hat{k}^\alpha (1 - \gamma_A) \]^\(1-\alpha\)

Next, write this in terms of growth rates (see the footnote on page 196 for an example):
Notice that the \((1 - \gamma_A)\) drops out, since the fraction of workers in production is assumed to be fixed. Next, notice that if we divide equation 2) by \(k\), we get the growth rate of \(k\):

\[
6) \quad \frac{\Delta k}{k} = k = \gamma_y \left( \frac{y}{k} \right) - \delta
\]

So, we can just plug in equations 3 (the growth rate of technology) and 6 (the growth rate of capital) into 5 (the growth rate of output) to get:

\[
7) \quad \hat{y} = \left( \frac{1}{\mu} \right) \gamma_A L + \alpha \left( \gamma_y \left( \frac{y}{k} \right) - \delta \right)
\]

O.K., so what? Well, we want to ask how fast this economy grows relative to one where there is no capital at all (the simple one in Chapter 8). First, appreciate the fact that now there is no real “steady-state” where \(\Delta k = 0\) (like in the traditional Solow model), because continuous technological growth allows for continuous capital growth.

Recall what an increase of technology in the Solow model looks like (see diagram below). Both production and investment shift up and out, and this gives you a higher level of steady-state capital per person. But of course, when \(A\) continuously grows, these shifts keep happening, and so the steady-state capital per person levels keep rising as well.
In other words, in this model, capital per person grows at a constant rate for ever:

\[ \hat{k} = \gamma_y \left( \frac{y}{k} \right) - \delta = x, \] where \( x \) is just some number. Notice then that in order for \( k \) to grow at a constant rate, \( \left( \frac{y}{k} \right) \) must be constant. And in order for this ratio to be constant, it must be that \( \hat{k} = \hat{y} \). So we can substitute in \( \hat{y} \) for equation 6), and rewrite our growth equation 7) as:

8) \[ \hat{y} = \left( \frac{1}{\mu} \right) \gamma_A L + \alpha \left( \hat{y} \right) \Rightarrow \]

\[ \hat{y} = \left( \frac{1}{1 - \alpha} \right) \left( \frac{1}{\mu} \right) \gamma_A L \]
The beautiful thing is that now we can directly compare the long run growth of this economy with the one where there was no capital at all. The long-run growth of that economy was just $\hat{Y} = \left( \frac{1}{\mu} \right)^{\gamma_A} L$ (this is equation 8.3 on page 214). Our equation 8) implies that this economy unambiguously grows faster, since $\alpha < 1$. Why? The growth in technology is complemented by growth in capital.

A few things to point out:

a) **Comparison of Growth Rates.** We now have three models to compare. The Solow model predicts that long-run growth is zero. The Romer model of section 8.2 in Weil predicts that long-run growth is positive but moderate. And finally, this model predicts that long-run growth is positive and quite rapid. Which of these do you think is the most plausible?

b) **The Role of $\alpha$.** As always, $\alpha$ illustrates the importance of capital in production. But there’s a big difference in the relationship between $\alpha$ and output in this model compared with the traditional Solow model. Recall that in Solow, we came up with a steady state level of output per person as a function of $A$, the savings rate, and the depreciation rate (see equation 3.3 on page 63). A higher $\alpha$ meant that steady-state income would be higher (increases in investment would increase income even more, while increases in depreciation would decrease income even more – in other words, capital has become more important). But nothing happens to long-run growth – Solow predicts that if $\alpha < 1$, diminishing returns to capital means that growth will ultimately slow down to zero.

Notice that in this model, on the other hand, the bigger is $\alpha$, the faster is long-term growth. This is the lesson of equation 8). Think of the logic behind this result.

c) **Changes in “Savings Rates.”** As mentioned before, this model really implies two ways to save for the future. You could either consume less output and save more of it (increase in $\gamma_Y$), or you could have less people produce stuff and more people research (increase in $\gamma_A$). What would be the short term affects of each change? What would be the long term affects of each change.

In short, increasing $\gamma_Y$ will produce faster growth in the short run but will leave long-run growth unchanged (this was the lesson from Solow, except long-run in that case was just zero). Increasing $\gamma_A$ on the other hand will produce permanently higher growth rates (this was the lesson from Chapter 8).
Increase in $\gamma_Y$

Both changes get you more “stuff” (higher $y$) – but the first case produces just a permanent level effect, while the second produces a permanent growth effect.