Optimal Control of Dynamic Pulse Power Loads in Naval Power Systems Using the Pontryagin Minimum Principle and Dynamic Programming

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Abstract—This paper addresses the electrical power management control problem for naval vessels. Traditional electrical system designs use a small number of large generators to provide sufficient power. Modern and future ships have a variety of power sources and loads, including fast pulse power loads. These various power sources and sinks must be coordinated to attain maximum performance. The analytical Pontryagin minimum principle is used to produce solutions for a simple system model and provide insight into the problem. A numerical optimization technique, Deterministic Dynamic Programming, is then used to develop similar solutions and study similarities and differences between the two techniques.

I. INTRODUCTION

Naval vessels must generate and deliver power to a variety of loads. Traditionally the largest power load by far is ship propulsion, which is mechanically supplied by steam turbines or gas turbine engines through a reduction gear. The ship electrical system services the radar, communication, and “hotel” loads of heating, lighting, etc. Power is supplied by a small number (2-4) of traditional synchronous generators.

An increasing proportion of modern naval vessels use electric propulsion to allow better fuel efficiency, more flexible power distribution, and damage survivability. Advanced technologies like electric catapults, radars, lasers, and rail guns impose significant pulsed loads on the power system. These technologies are expected to be deployed over the next few decades and will rival the propulsion electric load in power demand.

With these developments, traditional system design concepts become infeasible. It is generally not practical to size traditional generation to meet the worst-case simultaneous pulse power loads. Instead, the power system is comprised of a mix of traditional generators, power converters, and power storage elements. These various power sources and loads are then coordinated to satisfy the ship’s demands as effectively as possible, with the understanding that it may not be feasible to operate all the equipment at one time.

Although a power grid is electrical by nature, a large portion of the control problem reduces to coordinating the mechanical dynamics of various actuators. Gas turbines, diesel engines, and propulsion power trains typically have much slower dynamics than the electrical system and dominate the system dynamics.

This paper addresses the Power Management Control (PMC) problem of coordinating various sources and loads. The focus is on high-level power coordination and scheduling, which largely reduces to a power balance problem. The electrical details of designing and operating a power system in such a dynamic environment are quite challenging, but are ignored here.

The basic attributes of the PMC problem are studied using the Pontryagin Minimum Principle, an analytical technique. Although such analytical techniques may not be practical for a real system, they provide basic insight into the problem and highlight characteristics of optimal solutions. These techniques can also identify configurations or operating points that may be difficult to solve or have multiple solutions.

Although largely analytical, solutions using the minimum principle require solving for one or two numerical parameters. The search space for these parameters is shown to be highly non-convex; typical gradient-descent solvers do not work. A branch and bound numerical technique is shown to produce useful results, although relatively small changes in operating point can cause the solver to fail. This lends insight into which operating conditions may be challenging to solve for any algorithm.

Solutions are also computed using Deterministic Dynamic Programming (DDP) [1], [2]. This technique is more flexible for dealing with a real system, but does not always provide the same intuitive insight as analytical techniques. The DDP results show similar characteristics to the analytical results, but also demonstrate some of the pitfalls that arise using an inherently numerical algorithm rather than analytical techniques.

II. PROBLEM SETUP

A. Model

The system considered here has two power sources: a prime mover with relatively slow dynamics connected to a generator, and a power converter with fast dynamics. This power converter may be the bidirectional propulsion motor drive for a ship, or some other bidirectional power source. The power converter dynamics are assumed to be faster than the prime mover and load dynamics and are treated as instantaneous.

The prime mover mechanical power output $x_{PM}$ is modeled as a first-order response to the bounded command $u_{PM}$ with time constant $\tau$,

$$\dot{x}_{PM} = \frac{1}{\tau}(u_{PM}(t) - x_{PM}(t)).$$

(1)

The power converter is assumed to exactly produce the commanded power $u_{conv}(t)$. The system must track a reference power demand $r(t)$. Any power not provided by the converter
or the prime mover is assumed to come from the inertial energy in the generator and cause an AC frequency droop. This deviation in power tracking $\hat{P}$ can be quantified as

$$\hat{P}(t) = P_M(t) + u^\text{conv}(t) - r(t).$$  \hspace{1cm} (2)

The deviation of the AC bus frequency $\omega$ from its nominal set point $\bar{\omega}$ can be expressed as $\omega = \bar{\omega} - \tilde{\omega}$ and is modeled as the total energy supplied or absorbed

$$\tilde{\omega} = \hat{P}(t).$$  \hspace{1cm} (3)

The generator is relatively slow but has a large power capability, while power converter is very fast but has limited power output capability. This forces a clear compromise between the two power sources and optimal solutions will use the power sources together to utilize each of their strengths.

B. Optimization Formulation

The primary optimization goal is to minimize frequency deviations while tracking the reference power command. A secondary goal is to minimize the generator ramp rate $\dot{x}_{PM}$. This second goal is straightforward. Using an integral cost function, a quadratic penalty is applied to (1) to yield a cost $\left( u_{gen}(t) - x_{PM}(t) \right)^2$ by omitting the scaling constant $\tau$.

Including the frequency deviation in the cost function presents two choices. The first is to strictly penalize the power mismatch using a cost $\hat{P}(t)^2$. This is the simplest formulation as it omits the requirement to track the additional state $\omega$. The second option is to include the frequency state $\tilde{\omega}$ and include a cost $\tilde{\omega}^2$.

These two optimization formulations may be expressed as

**Formulation 1:**

$$\min \int_0^T \hat{P}(t)^2 + k(u_{PM}(t) - x_{PM}(t))^2 \hspace{1cm} (4)$$

subject to,

$$\dot{x}_{PM} = \frac{1}{\tau}(u_{PM}(t) - x_{PM}(t))$$

$$u_{PM}(t) \in [-1 \hspace{0.2cm} 1]$$

$$u^\text{conv}(t) \in [-0.2 \hspace{0.2cm} 0.2]$$

**Formulation 2:**

$$\min \int_0^T \tilde{\omega}(t)^2 + k(u_{PM}(t) - x_{PM}(t))^2 \hspace{1cm} (5)$$

subject to,

$$\dot{x}_{PM} = \frac{1}{\tau}(u_{PM}(t) - x_{PM}(t))$$

$$\omega = \frac{1}{I_w}\hat{P}(t)$$

$$u_{PM}(t) \in [-1 \hspace{0.2cm} 1]$$

$$u^\text{conv}(t) \in [-0.2 \hspace{0.2cm} 0.2]$$

The frequency deviation $\tilde{\omega}$ is scaled by the system inertia $I_w$.

From a practical aspect, Formulation 2 directly penalizes the time integral of frequency deviations. Formulation 1 penalizes the time integral of the derivative of those deviations, which is somewhat akin to penalizing the maximum deviation.

### III. Optimization via Pontryagin Minimum Principle

This section studies the optimization problem using the Pontryagin minimum principle [3], an analytical technique which is generally not practical for real-time implementation. The goal is to gain insight into basic fundamental issues, and to obtain reasonable solutions that can be used to verify more complex numerical methods developed later. Numerical optimization techniques for this type of problem will typically produce a solution, but it can be difficult to determine the quality of that solution and if that solution is locally or globally optimal, if at all. Analytical techniques can also yield basic insight into characteristics of good solutions and the underlying reasons for certain behaviors, both of which are useful in later stages.

The Pontryagin minimum principle is based on the calculus of variations and uses something similar to Lagrange multipliers from a standard optimization problem. It provides a set of necessary conditions for optimality, but those conditions are not sufficient. Essentially, these conditions provide an easy way to obtain characteristics of an optimal solution, but in general do not directly yield that solution. With careful problem setup and analysis, this method can yield an exact optimal solution for certain classes of problems.

#### A. Similarities to Hybrid Electric Vehicle Control

This analysis was inspired by the energy management problem for hybrid electric vehicles, which is a similar optimization problem. In that case, a few simplifying assumptions allow the use of the minimum principle to directly determine optimal control solutions. This method is termed the Equivalent Consumption Minimization Strategy (ECMS) [4], [5], [6], [7], [8], [9], [10]. While more sophisticated techniques are also used for optimization, the ECMS technique provides excellent performance under many conditions. More importantly, it provides fundamental insight into the problem. The ECMS technique dictates that the optimal control $u^*_f(x)$ is a compromise between fuel usage $m_f(x, u)$ and the change in battery State of Charge $\Delta SOC(x, u)$,

$$u^*_f(x) = \arg\min_{u \in U}[m_f(x, u) + \lambda_k \Delta SOC(x, u)].$$  \hspace{1cm} (6)

The change in battery charge is equivalent to the fuel consumption by an equivalence factor $\lambda$, hence the ECMS name. The parameter $\lambda$ is constant in time under relatively simple assumptions. While the exact optimal solution is non-causal, the form of 6 provides direct insight into the nature of the problem and works quite well even with causal estimates of $\lambda$.

This is a surprisingly powerful result and reduces a complex problem down to an intuitive explanation. This section attempts to gain similar insight into the optimization problem considered here.

#### B. Solution Techniques

This section considers Formulation 1 of the optimization problem. If generator ramp rate is neglected ($k=0$ in (5)) and the reference power $r(t)$ can be exactly tracked with the
existing actuators, the optimal solution is non-unique because there are various combinations of the two power sources that yield zero cost. When the derivative of the generator power is included in the cost function, solutions are likely to be unique. This is a reasonable cost from a physical standpoint in order to minimize transients on the generator.

The Pontryagin minimum principle is used to determine the form of the optimal control. The fundamental idea is that a separate dynamical system, the “costate,” is used to quantify the compromise between current and future cost.

The Hamiltonian of the system is

$$ H = \dot{P}(t)^2 + k(u_{PM}(t) - x_{PM}(t))^2 + \frac{\lambda}{\tau}(u_{PM}(t) - x_{PM}(t)). $$  (7)

The dynamics of the costate $\lambda$ evolve as

$$ \dot{\lambda} = \nabla_x H = -2\dot{P}(t) + 2k(u_{PM}(t) - x_{PM}(t)) + \frac{\lambda}{\tau}. $$  (8)

For any optimal control the Hamiltonian (7) has a global minimum with respect to the control. The power converter command $u_{conv}(t)$ only affects the power mismatch $\dot{P}$ in (7), so the command is

$$ u_{conv}(t) = \arg\min_{u \in U} \dot{P}(t)^2 = r(t) - x_{PM}(t). $$  (9)

The prime mover command $u_{PM}(t)$ is more complex,

$$ u_{PM}(t) = \arg\min_{u \in U} H = -\frac{\lambda}{2k\tau} + x_{PM}(t). $$  (10)

These are necessary but not sufficient conditions for optimality. In principle, a large number of controls can yield optimal solutions, and there may be a variety of control laws (not all optimal) that satisfy the conditions. By carefully structuring the cost function, these necessary conditions can be used to dictate control laws (9) and (10). Note that with a zero penalty $k$ on prime mover rate, the control command (10) becomes undefined. Penalizing the rate has a physical justification; unnecessarily rapid changes in power can cause equipment wear and increased fuel burn.

The state $x$ was assigned a fixed initial and final state of zero over a finite time interval of 10 s. Obtaining a solution requires solving the two-point boundary value problem for the state and costate. This can be quite tricky, as shown later. The most successful technique exploits the simplicity of this problem in that the final state is monotonic in the initial value of the costate. A branch-and-bound approach is used by guessing the initial value of the costate, integrating both sets of dynamics forward in time, and determining the final value of the state. The two state values that bound the target are used to determine a better guess for the initial value of the costate.

C. Results

This optimization technique is capable of generating optimal trajectories that look reasonable and exhibit many of the characteristics one would expect. The system trajectories for a decaying pulse load are shown in Figure 1a.
The traces show the Target Power delivered to the load (black) which is a step load of 0.48 which then decays slowly. The prime mover power output (green) lags behind the prime mover command (red) due to its slow dynamics. The power converter command (light blue) is limited at +/-0.2. The total Actual Power (blue) supplied is the sum of the prime mover and drive powers. As this is an AC system model, any power mismatch between the source and load powers is absorbed or supplied by the synchronous generator rotational inertia. This Inertial Power (purple) is roughly equivalent to the rate of system frequency deviation and is the difference between the Target Power and the Actual Power.

The traces show some interesting characteristics. The pulse power load is clearly faster than the prime mover capability, but requires more power than the power converter can provide. The prime mover is slowly ramped up to roughly half the pulse power load, and the converter is used to balance out this power and maintain system frequency. This serves two purposes. The prime mover can be brought up to a relatively high power level to provide pulse power support, while the converter is providing full negative power and can switch fully rail-to-rail to provide the highest rate of power change. Even so, the pulse amplitude (0.48) is larger than the full range of the converter (0.4) and some of the power comes from the inertial energy store.

As previously mentioned, a major component of the optimization technique is the costate $\lambda$, shown in Figure 1b. Intuitively, this state represents future information about the cost that will be incurred. In this case, it dictates the idea prime mover rate of change. The negative values (-6) before the pulse dictate a relatively fast ramp up, while the positive values after the pulse (+3) dictate a slower ramp down.

The next case studied is a pure pulse load of 0.48 with no decay. These are the same dynamics and cost function as the previous case, but with a different load profile. The results are shown in Figure 2 with the same traces as used previously.

The results in Figure 2 show many of the same characteristics as those in Figure 1. The prime mover is slowly ramped up to roughly half of the pulse value, while the power converter provides balancing power to maintain frequency. The AC frequency (Inertial Power) is stored up slightly before the pulse, dips during the pulse, and recovers. The prime mover exhibits mostly constant ramp rates.

The most striking aspect of these results is the symmetry of the response. The system is designed to handle a pulse and then recover. The inherent symmetry in the underlying dynamics, cost function, and start and end times dictate that the post-pulse dynamics are basically mirror images of the pre-pulse dynamics.

The evolution of the costate $\lambda$ is shown in Figure 2b. The symmetry of the results is also visible in this figure. The negative values (-0.004) before the pulse dictate the ramp up, while the opposite positive values after the pulse (0.004) describe an identical ramp down.

D. Discussion

Under certain conditions, this technique can produce nearly-perfect optimization results that show smooth traces and exact
convergence, as one might hope when using an analytical technique. However, as formulated here, this technique suffers from a major drawback in the form of numerical solution accuracy. Obtaining solutions is highly dependent on the numerical parameters involved, and the numerical integration of the ODEs must be much more accurate than typically used.

This sensitivity is likely a result of the solution method selected. Specifically, picking initial conditions to yield a final state after integrating forward in time is numerically sensitive, especially for "long" time horizons. This problem is exacerbated by the nature of the costate here: it is an unstable system. In some sense, the costate represents the future cost that will be incurred based on the state. As one travels backward through time and moves further from the costly event, the relative importance of a state is reduced. This manifests itself as a decaying system in reverse time, which is an unstable system in forward time. For future work, this technique may be better served by treating this as a true boundary value problem and using an appropriate solver.

Figure 3 shows the relative importance of numerical solution accuracy for the solution shown in Figure 1. The horizontal axis is the initial guess for the costate, which exactly determines the solution. The vertical axis is the total cost of the solution. Two curves are shown that are generated for different values of the relative ODE integration tolerance, $10^{-13}$ (high accuracy) and $10^{-3}$ (low accuracy). The large difference between the two implies that this solution method is sensitive to the solution accuracy.

The optimization attempts to minimize the total cost. If the problem is framed as “minimize total cost by varying the costate,” most numerical solutions are based on some form of gradient-descent algorithm. For a non-smooth function (like the solid lines in the figure) these algorithms may fail and find a local minimum that is not the true global minimum. For the particular system studied here, there is a relatively large region of very low cost, implying that there is some potential slop in the solution and there is a group of related controls that generate similar total cost. Physically, this arises because the converter can exactly balance the prime mover power within a certain range, leaving the same total output power.

Another interesting feature is the large square peak in the high accuracy solution. The low accuracy solution completely missed this phenomenon. This feature also hints that even the high-accuracy solution may not provide an easy solution under all conditions.

Figure 4 shows an example of this problem: although an optimal solution may exist, the numerical techniques used here cannot find the solution under some conditions. For this exercise, the target power is a pulse command of 0.5, which is only marginally larger than the 0.48 pulse command used earlier. The branch and bound converged to two guesses for the initial costate value that differ by 8.6736e-019, which is 2x the best floating point precision of Matlab. The final values of the state for these two guesses were -0.1945 (shown below in Figure 4) and 0.8846. This means that the required numerical accuracy of the initial guess using this method is better than double-precision floating point.

For the target power shown in Figure 4, the available search space for the initial value of the costate is shown in Figure 5. From an optimization perspective, this figure shows the Total Cost for various initial guesses for lambda. This figure reinforces the inherent difficulty in finding a solution for this case; even the curve using highly accurate ODE solutions demonstrates shows many local minima, very sharp transitions, and almost discontinuous behavior, including a
sharp “false” local minimum just to the left of the main global minimum. This type of problem is very difficult to solve using conventional minimization techniques.

IV. DYNAMIC PROGRAMMING

This section studies the use of Deterministic Dynamic Programming (DDP) to solve the optimization problem. DDP is a particular variant of general Dynamic Programming techniques which covers a broad class of stochastic and deterministic algorithms [1], [2].

The DDP algorithm assumes a known future and system dynamics in order to exactly calculate optimal solutions. Although future knowledge is used in the calculation, the result can be implemented as a time-dependent feedback controller in order to correct for small variations.

A. Solution Techniques

The general DDP algorithm [1], [2] minimizes a cost \( c_k(x_k, u_k) \) over a finite horizon \( T \) and is briefly summarized here. The general optimization problem solved by DDP is

\[
\min_{u_k} \sum_{k=0}^{T} c_k(x_k, u_k) \tag{11}
\]

subject to the system dynamics,

\[
x_{k+1} = f_k(x_k, u_k) \tag{12}
\]

with \( u_k \in U(x_k) \), where

\[
U_k(x_k) = \{ u_k \mid g_k^1(x_k, u_k) \leq 0, \ g_k^2(x_k, u_k) = 0 \}.
\]

Actuator limits, power delivery bounds, and other system requirements are incorporated in the constraints \( g_k^1 \) and \( g_k^2 \), which are enforced at each time step. The cost function, dynamics, and constraints can all be time-varying as denoted by the subscript \( k \). This general setup covers the specific Formulations 1 and 2 of Section II-B.

The general DDP algorithm to solve (11) involves two parts: an off-line step that solves the optimization problem, and an on-line step to implement it. The off-line solution works backward in time to find the solution, while the on-line step uses this precomputed solution and can be used in a causal implementation.

The off-line solution step starts at the final time \( T \) and recursively steps backward in time solving

\[
J_k^*(x) = \min_{u_k} \left[ c_k(x_k, u_k) + J_{k+1}^*(f_k(x_k, u_k)) \right], \tag{13}
\]

where \( J_k^* \) is the optimal “cost to go” function which predicts the future cost of moving from the current state and time to the final time \( T \). The cost at the final time \( J_T(x) \) can be assigned by the designer.

The on-line implementation uses the precomputed \( J_k^*(x) \) to find an optimal control \( u^* \) that satisfies

\[
J_k^*(x) = \min_{u_k} \left[ c_k(x_k, u_k) + J_{k+1}^*(f_k(x_k, u_k)) \right]. \tag{14}
\]

The off-line solution to (13) can produce an optimal control trajectory \( u_k \), but for practical implementation the on-line calculation (14) acts as a feedback controller and computes \( u_k \) based on the current state. This feedback corrects for small variations in the system. Computing (14) in real-time is relatively simple, although the off-line calculation of (13) is much more difficult.

The DDP technique is applied to Formulation 2 of the optimization problem (6) using the previously described cost functions and system modes.

B. Results

The DDP solution to Formulation 2 of the optimization problem are shown in 6 for a moderate penalty on prime mover ramp rate. These results show similar optimal characteristics to the previous analytical results, but also demonstrate some of the difficulties of using this method.

Overall, the defining system response characteristics are identical to those shown in Figure 2. The prime mover ramp rate controls are shown in Figure 4. The prime mover is slowly ramped up to roughly half of the expected pulse power, the power converter saturates on its negative rail before the pulse, and then switches fully rail to rail to try and track the pulse.

The frequency of the AC system is included explicitly as a state in this formulation, so the results show system frequency rather than system power.

There are several features that highlight the differences between the two techniques. The control commands in this case are discretized into finite steps of 0.1 for the prime mover as visible in the figure. Discretization is also responsible for the slightly jagged converter command and frequency response. The solution is also discretized in time, so the pulse
itself is not an instantaneous step, but rather occurs in one time period of 0.1 s. The prime mover does not return to zero power because there is no imposed constraint on the final state as there was before. This can be added if desired, but this solution is somewhat like what would happen in preparation for an immediately following pulse.

V. CONCLUSIONS

Analytical optimization techniques were applied to a simple model of a ship power system. These analytical techniques provide basic insight into the nature of solutions, but they can be difficult to use under some conditions due to numerical issues when solving for parameters.

A numerical implementation of Deterministic Dynamic Programming was developed to move towards a practical solution that may be used in a real system. The results showed similar characteristics to the analytical techniques and revealed some drawbacks of the numerical optimization.

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REFERENCES


