Knowledge Demonstration Opportunity 1: SM221, Calculus III

Name: **ALLMAN**

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Read all of the following information before starting:

- You are allowed pencils, pens, your TI-36X calculator, and your wits. That is all. In particular, no computers, notes, books, smartphones, iPads, or pocket-sized hobbits.

- To receive full credit, justify your work clearly and in order. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your “final” answer is correct).

- Use sentences to explain your reasoning. Please keep written answers brief; and simultaneously clear!

- Box or otherwise indicate your final numeric answers.

- Good luck!

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Problem 1 (10 points). Let \( f(x, y) = 19x^2 - 3x^2y - 7xy^2 + 14y^3 \).

(a) Compute the gradient \( \nabla f(x, y) \)

(b) Compute the directional derivative \( D_v f(1, -1) \) in the direction \( v = \left(-\frac{3}{5}, \frac{4}{5}\right) \).

(c) Write an equation for the tangent plane to the graph \( z = f(x, y) \) at the point \( (1, -1, 1) \).

\[
\nabla f = \left< 38x - 6x y - 7y^2, -3x^2 - 14xy + 42y^2 \right>
\]

\( \nabla f(1, -1) = \left< 37, 53 \right> \)

\[
D_v f(1, -1) = \nabla f(1, -1) \cdot v = \left< 37, 53 \right> \cdot \left( -\frac{3}{5}, \frac{4}{5} \right) = \frac{53 \cdot 4 - 37 \cdot 3}{5} = \frac{212 - 111}{5} = \frac{101}{5}
\]

\( f_x(1, -1) = 37 \quad f_y(1, -1) = 53 \quad f(1, -1) = 1 \)

\[
2 = 1 + 37(x - 1) + 53(y + 1)
\]

\[
0 = 37x + 53y - 2 + (1 - 37 + 53)a
\]
Problem 2 (10 points). Answer each of the following multiple choice questions.

(a) Suppose that $T(x, y, z) = \frac{4x + 2y^2}{z+1}$ represents the temperature in degrees celcius in the first octant (aka where $x > 0$, $y > 0$, and $z > 0$) in 3D space. At the point $(2,1,2)$ which of the following vectors is in parallel to the direction of greatest instantaneous change in the temperature?

(i) $(4, 4, 10)$;
(ii) $(4, 4, -10)$;
(iii) $(4, 4, -10/3)$;
(iv) $(4, 4, 10/3)$;
(v) $(-4/3, -4/3, -10/9)$.

\[ \nabla T = \left\langle \frac{4}{z+1}, \frac{8y}{z+1}, -\frac{4x + 2y^2}{(z+1)^2} \right\rangle \]

\[ \nabla T \bigg|_{(2,1,2)} = \left\langle \frac{4}{3}, \frac{4}{3}, -\frac{8+2}{9} \right\rangle = \left\langle \frac{4}{3}, \frac{4}{3}, -\frac{10}{9} \right\rangle \]

(b) Which of the following best describes the path in 2D space which is parameterized by the function below?

\[ \mathbf{r}(t) = \left\langle 10t \sin(t^2), 10t \cos(t^2) \right\rangle, \quad t \geq 0 \]

(i) a spiral whose radius increases as $t$ increases;
(ii) an ellipse whose axes increase as $t$ increases;
(iii) a circle around which we move faster and faster as $t$ increases;
(iv) a quarter of a circle in the first quadrant;
(v) a parabola on which we move back and forth.

Since $x^2 + y^2 = (5\cos(t^2))^2 + (5\sin(t^2))^2 = 25$, we know its a circle.

(c) The first picture below is the contour plot of a function $z = f(x, y)$. The next two plots are graphs of functions $x(t)$ and $y(t)$. Use them to decide if the derivative $\frac{dz}{dt}_{t=1/2}$ is

(i) positive;
(ii) negative;
(iii) approximately zero;
(iv) can't tell from the information given.

\[ \frac{dz}{dt} = \frac{dx}{dt} + \frac{dy}{dt} \]

\[ \frac{dx}{dt} \bigg|_{t=1/2} = 0 \]

\[ \frac{dy}{dt} \bigg|_{t=1/2} > 0 \]

\[ t = \frac{5}{2} \Rightarrow x = 1, \quad y = 0 \]
Problem 3 (10 points). In class we only discussed quadric surfaces whose defining equations had no terms of the form: $xy$, $yz$, or $xz$. However, because each of these monomials is of degree two, they still can be used to define quadric surfaces. In this problem, consider the quadric surface defined by the equation

$$y^2 = xz$$

(a) Sketch traces for the values $x = 0$, $x = 1$ and $x = -1$ below:

(b) Sketch traces for the values $y = 0$, $y = \pm 1$ and $y = \pm 2$ below:
(c) Identify which picture corresponds to the quadric surface cut out by $y^2 = xz$. 
Problem 4 (10 points). Suppose that an ion travels through an electric field in 3D space with acceleration at time \( t \geq 0 \) given by the vector function
\[
a(t) = (\sin(t), 2 \cos(t), \sin(\theta t)).
\]

(a) If its initial velocity is \( \mathbf{v}(0) = (-1, 0, -1/6) \) and its initial position is the origin, find parametric expressions for the velocity \( \mathbf{v}(t) \) and position \( \mathbf{r}(t) \).

\[
\mathbf{v}(t) = \left< -\cos t, \ 2\sin t, \ -\frac{1}{6} \cos (6t) \right> + \left< c_1, c_2, c_3 \right> = \left< -1, 0, -\frac{1}{6} \right> \text{ given.}
\]

\[
\Rightarrow \ c_1 = c_2 = c_3 = 0 \quad \text{So...}
\]

\[
\mathbf{v}(t) = \left< -\cos t, \ 2\sin t, \ -\frac{1}{6} \cos (6t) \right>
\]

\[
\mathbf{r}(t) = \left< -\sin t, \ -2\cos t, \ -\frac{1}{36} \sin (6t) \right> + \left< d_1, d_2, d_3 \right>
\]

\[
\mathbf{r}(0) = \left< 0, -2, 0 \right> + \left< d_1, d_2, d_3 \right> = \left< 0, 0, 0 \right> \text{ given}
\]

\[
\Rightarrow \ d_1 = d_3 = 0 \quad \text{&} \quad d_2 = 2 \quad \text{So...}
\]

\[
\mathbf{r}(t) = \left< -\sin t, \ -2\cos t + 2, \ -\frac{1}{36} \cos (6t) \right>
\]

(b) Write down, but do not evaluate, an integral which computes the arc-length of the particles path from time \( t = 0 \) to time \( t = 3\pi \).

\[
s = \int_0^{3\pi} \left| \mathbf{r}'(t) \right| \, dt = \int_0^{3\pi} \left| \mathbf{v}(t) \right| \, dt
\]

\[
= \int_0^{3\pi} \sqrt{\cos^2 t + 4 \sin^2 t + \frac{1}{36} \cos^2 (6t)} \, dt
\]