Knowledge Demonstration Opportunity 2: SM221P, Calculus III

Name: ______________________

4 October 2017

Read all of the following information before starting:

- You are allowed pencils, pens, your TI-36X calculator, and your wits. That is all. In particular, no computers, notes, books, smartphones, iPads, or pocket-sized hobbits.

- To receive full credit, justify your work clearly and in order. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your “final” answer is correct).

- Use sentences to explain your reasoning. Please keep written answers brief; and simultaneously clear!

- Box or otherwise indicate your final numeric answers.

- Good luck!

<table>
<thead>
<tr>
<th>Problem</th>
<th>Possible</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>40</td>
<td></td>
</tr>
</tbody>
</table>
Problem 1 (10 points). Let \( f(x, y) = \ln (9 - x^2 - y^2) \).

(a) Describe (with correct inequalities, or a drawing, or words) the domain and range of \( f \).

(b) Compute the partial derivatives \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \).

(c) Compute the value of the directional derivative \( D_v f(2, 1) \) with \( v \) in the direction of \( \langle 3, -4 \rangle \).
Problem 2 (10 points). Each of the pictures below is a contour plot for some function $f(x,y)$. Use the plots to answer the questions below. Circle your final answer.

(a) In contour plot III, suppose that the positive $y$-direction is north and the positive $x$-direction is east. Which of the following most accurately describes the direction of the gradient $\nabla f(0,0)$?

(i) north  
(ii) south  
(iii) east  
(iv) west

(b) For which of the contour plots is the partial derivative $f_y(1,1)$ positive?

(i) I and II  
(ii) III and IV  
(iii) all of these  
(iv) none of these

(c) Which of the following numbers is larger?

(i) $f_{yy}(0,-1.5)$ on IV  
(ii) $f_{xx}(-1,1)$ on III  
(iii) neither, they are roughly equal
Problem 3 (10 points). At time $t = 0$, a particle is at the point $(0, -1, 0)$ and has an initial velocity of $(1, 0, 1)$. Moreover, for all $t \geq 0$, the particle experiences an acceleration

$$a(t) = \langle -\sin(t), \cos(t), \frac{1}{2}(t + 1)^{-1/2} \rangle.$$

(a) Compute parameterizations for the velocity and position vectors of the particle (i.e. find vectors $v(t)$ and $r(t)$).

(b) Use your answer from (a) to compute the arclength of the particle’s path from $t = 2$ to $t = 7$. 
Problem 4 (10 points). For \(0 \leq u \leq \pi\) and \(0 \leq v \leq \pi/4\), consider the parameterized surface in \(\mathbb{R}^3\) given by
\[S(u, v) = (3 \cos(u) \sin(v), 3 \sin(u) \sin(v), 3 \cos(v)).\]

(a) Identify the image below which depicts the range of the function \(S(u, v)\) [HINT: recall that the spherical coordinates in \(\mathbb{R}^3\) satisfy the equations \(x = \rho \cos \theta \sin \phi\), \(y = \rho \sin \theta \sin \phi\), \(z = \rho \cos \phi\)].

(b) Suppose that the temperature in the region of space near the surface \(S(u, v)\) is given (in \(\text{C}^\circ\)) by the function \(T(x, y, z) = \frac{x}{\sqrt{x^2 + 1}}\). Use the chain rule to compute the rate of change in temperature with respect to \(v\) along the surface at the point corresponding to \(u = 0\) and \(v = 0\).