Knowledge Demonstration Opportunity 3: SM221, Calculus III

Name: \( \underline{\text{ALLMAN}} \)

29 March 2018

Read all of the following information before starting:

- You are allowed pencils, pens, your TI-36X calculator, and your wits. That is all. In particular, no computers, notes, books, smartphones, iPads, or pocket-sized hobbits.

- To receive full credit, justify your work clearly and in order. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your "final" answer is correct).

- Use sentences to explain your reasoning. Please keep written answers brief; and simultaneously clear!

- Box or otherwise indicate your final numeric answers.

- Good luck!

<table>
<thead>
<tr>
<th>Problem</th>
<th>Possible</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>40</td>
<td></td>
</tr>
</tbody>
</table>
Problem 1 (10 points). Compute the value of the triple integral $\iiint_E e^{(x^2+y^2+z^2)^{3/2}} \, dV$ where $E$ is the region in the first octant between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$.

Use spherical coordinates:

$$E = \left\{ 1 \leq \rho \leq 2, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq \frac{\pi}{2} \right\}$$

$$e^{(x^2+y^2+z^2)^{3/2}} \quad \rightarrow \quad e^{\rho^3}$$

$$dV \quad \rightarrow \quad \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

The integral is:

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 e^{\rho^3} \frac{1}{\rho} \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= \left[ \int_0^{\pi/2} \frac{1}{3} e^{\rho^3} \, d\rho \right] \cdot \left[ \int_0^{\pi/2} \sin \phi \, d\phi \right] \cdot \left[ \int_1^2 \rho^2 \, d\rho \right]$$

$$= \left[ \frac{\pi}{2} \right] \cdot \left[ -\cos \phi \right]_{0}^{\pi/2} \cdot \left[ \frac{1}{3} e^{\rho^3} \right]_{1}^{2}$$

$$u = \rho^3 \Rightarrow \, du = 3 \rho^2 \, d\rho$$

$$\rho = 1 \Rightarrow u = 1$$

$$\rho = 2 \Rightarrow u = 8$$

$$= \frac{\pi}{2} \cdot \frac{1}{3} (e^8 - e^1)$$

$$= \frac{\pi}{6} (e^8 - e)$$
Problem 2 (10 points). Answer each of the following multiple choice questions.

(a) The picture below shows a conservative vector field $\mathbf{F}$ and a portion of a curve $C$ from the point $P$ to the point $Q$. Assume that $\mathbf{F}$ is defined on all of $\mathbb{R}^2$ and that $C$ is a smooth connected path outside of the depicted region. The value of the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is

Since $\mathbf{F}$ is conservative,

\[ \int_C \mathbf{F} \cdot d\mathbf{r} \] is path indep. 

(i) positive; 
(ii) negative; 
(iii) zero; 
(iv) can’t tell from the given information

Here, we see $\mathbf{F}$ is like a wind "at your back".

(b) Suppose that $\mathbf{F} = (P, Q)$ is a vector field, defined and differentiable on all of $\mathbb{R}^2$, and that $C$ is a simple closed curve. If $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$, which of the following must be true?

To show I, recall $\int_C \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, d\mathbf{r} = 0$.

I. $\mathbf{F}$ is conservative
II. $\iint_D (Q_y - P_x) \, dA = 0$, where $D$ is the region enclosed by $C$.

(a) I only
(b) II only
(c) I and II
(d) neither I nor II

(c) Consider the vector field $\mathbf{F}$ depicted below, and the three line segments $C_1$, $C_2$, and $C_3$. Let

\[ I_1 = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} \quad I_2 = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \quad I_3 = \int_{C_3} \mathbf{F} \cdot d\mathbf{r} \]

and choose the correct order for the value of the integrals.

(i) $I_1 < I_2 < I_3$;
(ii) $I_1 < I_3 < I_2$;
(iii) $I_2 < I_1 < I_3$;
(iv) $I_2 < I_3 < I_1$;
(v) $I_3 < I_1 < I_2$;
(vi) $I_3 < I_2 < I_1$;

Along $C_2$, $\mathbf{F}$ is always "in your face".
Along $C_1$, $\mathbf{F}$ is always "at your back".
Along $C_3$, $\mathbf{F}$ is probably $\int_{C_3} \mathbf{F} \cdot d\mathbf{r} \approx 0$. 

Just look where $\mathbf{F}$ is "at your back" or "in your face".
Problem 3 (10 points). Let \( C \) be the unit circle in \( \mathbb{R}^2 \), oriented counter-clockwise. Compute the following line integrals.

(a) \[ \int_C (x^2 + y^2) \, xy^3 \, ds \]

Parameterizes \( C \)

\[ \frac{d}{dt} (\cos t, \sin t) = (-\sin t, \cos t) \]

\[ ds = \left\| \frac{d}{dt} (\cos t, \sin t) \right\| \, dt = \sqrt{\sin^2 t + \cos^2 t} \, dt = dt \]

\[ \Rightarrow \text{Integral} = \int_0^{2\pi} \left( \cos^2 t + \sin^2 t \right) \cos t \, \sin^3 t \, dt \]

\[ = \int_0^{2\pi} \cos^2 t \sin^3 t \, dt = \int_0^\pi u^3 \, du \]

\[ u = \sin t \]

\[ du = \cos t \, dt \]

\[ t = 0 \Rightarrow u = 0 \quad t = 2\pi \Rightarrow u = \pi \]

\[ = \frac{u^4}{4} \bigg|_0^\pi = \frac{\pi^4}{4} \]

(b) \[ \int_C (3x^2 - 2y, 5x - e^y) \cdot dr \]

\( C \) closed! So use Green's Theorem...

\[ Q_x - P_y = 5 - (-2) = 7. \]

\[ \text{Line Integral} = \iint_C 7 \, dA = 7 \text{ Area(inside disk)} \]

\[ = 7 \left( \frac{\pi \cdot 1^2}{2} \right) \]

\[ = \sqrt{7 \pi} \]
Problem 4 (10 points). Let \( \mathbf{F} = \left( 2xy, x^2 + 2yz, y^3 \right) \)

(a) Prove that \( \mathbf{F} \) is conservative.

\[
\text{curl} \ (\mathbf{F}) = \left\langle R_y - Q_z, \ P_z - R_x, \ Q_x - P_y \right\rangle \\
= \left\langle (2y) - (0), \ (0) - (0), \ (2x) - (0) \right\rangle = \left\langle 0, 0, 0 \right\rangle \\
\]

OR just do ... (ble, you need it for (L))

\[
f = \int P \, dx = x^2y + g(y, z) \\
f = \int Q \, dy = x^2y + y^2z + k(x, z) \\
f = \int R \, dz = ay^2z + k(x, y) \\
\]

(b) Evaluate the line integral \( \int_\Gamma \mathbf{F} \cdot d\mathbf{r} \) where \( \Gamma \) is the curve parameterized by \( \mathbf{r}(t) = (t^3 + t, t^4 - t^2, te^t) \) for \( 0 \leq t \leq 1 \).

By Fundamental Theorem of Line Integrals

\[
\int_\Gamma \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) \\
= f(2, 0, e) - f(0, 0, 0) \\
= [0 + 0] - [0 + 0] = 0 
\]