Chapter 3

Limits

In this chapter we will introduce the notion of limits, which we will use to compute derivatives in later chapters.

**Problem 3.1.** Consider the function $f$ defined by $y = f(x) = 2x + 1$, which you encountered in Chapter 1.

(a) State the domain of $f$, and sketch the graph of $f$.

(b) As the values of $x$ get closer to 0, is there a number that the corresponding $y$-values get closer to? If such a number exists, it is called the limit as $x$ approaches 0 of $f(x)$, and is denoted by $\lim_{x \to 0} f(x)$.

(c) Does $f(0)$ exist? If so, what is $f(0)$?

**Problem 3.2.** Consider the function $g$ defined by $y = g(x) = \frac{2x^2 + x}{x}$, which you also encountered in Chapter 1.

(a) State the domain of $g$, and sketch the graph of $g$. How is this graph similar to and how is it different from the graph of $f$ above?

(b) As the values of $x$ get closer to 0, is there a number that the corresponding $y$-values get closer to? That is, find $\lim_{x \to 0} g(x)$.

(c) Does $g(0)$ exist? If so, what is $g(0)$?

**Problem 3.3.** Consider the functions $f(x) = 2x + 1$ and $g(x) = \frac{2x^2 + x}{x}$, from above.

(a) Is it the case that $f(0) = g(0)$?
(b) Is it the case that \( \lim_{x \to 0} f(x) = \lim_{x \to 0} g(x) \)?

Note that we may be interested in limits where \( x \) approaches a number different from 0. The following definition generalizes our notion of limit from above.

**Definition 3.4.** We say the limit of \( f \) as \( x \) approaches \( a \) is equal to \( L \) if, as \( x \) gets closer to \( a \), the corresponding values of \( f(x) \) get closer to \( L \). We write \( \lim_{x \to a} f(x) = L \).

(This definition is slightly informal, but works well in Calculus. A more formal definition is used in the course Real Analysis.)

**Problem 3.5.** Now, let \( f(x) = \frac{x^2 - 9}{x - 3} \).

(a) Explain why \( f \) is not defined at \( x = 3 \).

(b) Find a simplified function which is equivalent to \( f \) at every point except \( x = 3 \). Call this function \( g \).

(c) What is \( g(3) \)?

(d) What is \( \lim_{x \to 3} g(x) \)?

(e) What is \( \lim_{x \to 3} f(x) \)?

**Problem 3.6.** Evaluate the following limits.

(a) \( \lim_{x \to -5} x^2 - 1 \)

(b) \( \lim_{x \to -5} \frac{x^2 + 6x + 5}{x + 5} \)

(c) \( \lim_{x \to -5} \frac{x^2 + 3x - 10}{x^2 - 25} \)

(d) \( \lim_{x \to -5} 8 \)

**Problem 3.7.** Let us summarize what we have noticed thus far about limits.

(a) If \( f(a) \) exists, how would you find \( \lim_{x \to a} f(x) \)?

(b) If \( f(a) \) does not exist, how would you find \( \lim_{x \to a} f(x) \)?

Note that the methods for computing limits you described above work for most functions you will encounter, but they are not always true. Think of them as guidelines.

**Problem 3.8.** Evaluate the following limits.
(a) \( \lim_{u \to 4} \frac{u + 2}{u^2 + u + 1} \)

(b) \( \lim_{t \to 4} \frac{t^2 - 16}{t - 4} \)

(c) \( \lim_{t \to -4} \frac{t^2 - 16}{t - 4} \)

(d) \( \lim_{s \to 9} \sqrt{s} \)

Problem 3.9. This problem will be a bit different from the previous ones, and is meant to solidify your understanding of limits. Let \( f(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \).

(a) Sketch a graph of \( f \) between \( x = -2 \) and \( x = 2 \).

(b) Looking at the graph or a table of values, is there a number that \( f(x) \) gets closer to as \( x \) gets closer to 0?

(c) Explain why \( \lim_{x \to 0} f(x) \neq 1 \) even though \( f(0) = 1 \).

Now that we have evaluated a number of limits using the guidelines we established above, the following problems investigate functions for which they simply do not work.

Problem 3.10. Let \( f(x) = \frac{1}{x^2} \).

(a) Sketch the graph of \( f \).

(b) As \( x \) gets closer and closer to 0, is there a number that the values of \( f(x) \) get closer and closer to?

(c) What does that say about \( \lim_{x \to 0} f(x) \)?

Problem 3.11. Let \( g(x) = \frac{1}{x} \).

(a) Sketch the graph of \( g \).

(b) As \( x \) gets closer and closer to 0, is there a number that the values of \( g(x) \) get closer and closer to?

(c) What does that say about \( \lim_{x \to 0} g(x) \)?

Problem 3.12. Let \( h(x) = \frac{|x|}{x} \).

(a) Sketch the graph of \( h \).
(b) As \( x \) gets closer and closer to 0, is there a number that the values of \( h(x) \) get closer and closer to?

(c) What does that say about \( \lim_{x \to 0} h(x) \)?

**Problem 3.13.** In the problem above, it can be useful to look separately at \( h(x) \) when \( x > 0 \) and when \( x < 0 \). Let \( h(x) = |x| / x \) as above.

(a) If \( x > 0 \), what is the single value that \( h(x) \) gets closer to as \( x \) gets closer to 0 (while remaining greater than 0)? This is called the **limit of \( h(x) \) as \( x \) approaches 0 from the right** and is written as \( \lim_{x \to 0^+} h(x) \).

(b) If \( x < 0 \), what is the single value that \( h(x) \) gets closer to as \( x \) gets closer to 0 (while remaining less than 0)? This is called the **limit of \( h(x) \) as \( x \) approaches 0 from the left** and is written as \( \lim_{x \to 0^-} h(x) \).

**Problem 3.14.** Let \( w(x) = \begin{cases} x^2 & \text{if } x \leq 1, \\ 2x + 1 & \text{if } x > 1. \end{cases} \)

This is the same function we saw in Chapter 1.

(a) Compute \( \lim_{x \to 1^+} w(x) \).

(b) Compute \( \lim_{x \to 1^-} w(x) \).

(c) Explain why \( \lim_{x \to 1} w(x) \) does not exist.

There are many properties of limits that will help us both now with the computation of limits as well as later in the development of the derivative. We have actually been using these properties for some time without explicitly stating them. We will summarize these properties in the following theorem.

**Theorem 3.15.** Suppose that

\[
\lim_{x \to a} (f(x)) = L\, \text{ and } \lim_{x \to a} (g(x)) = M.
\]

Then

(a) \( \lim_{x \to a} (f(x) + g(x)) = L + M, \)

(b) \( \lim_{x \to a} (f(x) - g(x)) = L - M, \)

(c) \( \lim_{x \to a} (f(x) \cdot g(x)) = L \cdot M, \)
(d) \( \lim_{x \to a} \left( \frac{f(x)}{g(x)} \right) = \frac{L}{M} \), as long as \( M \neq 0 \),

(e) for any real number \( k \), \( \lim_{x \to a} (k \cdot f(x)) = k \cdot L \).

**Problem 3.16.** Suppose you know that \( \lim_{x \to 3} f(x) = 5 \) and \( \lim_{x \to 3} g(x) = -2 \). Evaluate the following limits.

(a) \( \lim_{x \to 3} (f(x) + g(x)) \)

(b) \( \lim_{x \to 3} (3f(x) - 4g(x)) \)

(c) \( \lim_{x \to 3} (f(x) + f(x)g(x)) \)

(d) \( \lim_{x \to 3} \left( \frac{f(x)}{g(x)} \right) \)

(e) \( \lim_{x \to 3} \left( \frac{g(x)}{f(x)} \right) \)