Notes on the basics of numbers

**Natural Numbers**

So far we have been working with numbers without stating any formal definitions or axioms about them. But as we saw when we tried to talk about one set having “fewer” elements than another, we do need to be careful about what we mean.

A good place to start is with a set of axioms that define the natural numbers. These axioms were first formulated by Giuseppe Peano in 1889. Peano grew up on a farm in northwestern Italy; he had an uncle who was a priest and lawyer in Turin who noticed he was a smart kid, and brought him to Turin for secondary school, and later university. He earned his doctorate in mathematics and spent his entire career at the University of Turin. [For more information, see the MacTutor History of Mathematics archive, http://www-history.mcs.st-and.ac.uk.]

The following five axioms give the properties of a set \( \mathbb{N} \) whose elements we will call natural numbers. These properties define the set. The undefined terms are “1”, “(natural) number”, and “successor”.

**Peano Axioms for \( \mathbb{N} \).** You can compare the next five axioms to those given in our book on page 65.

**Axiom 1** (PA1). 1 is a natural number.

**Axiom 2** (PA2). Every natural number has a unique successor which is a natural number.

**Axiom 3** (PA3). No two different natural numbers have the same successor.

**Axiom 4** (PA4). There is no natural number whose successor is 1.

**Notation.** If \( n \in \mathbb{N} \) we denote its successor by \( n' \).

**Axiom 5** (PA5, a.k.a. The Principle of Mathematical Induction). If

1. \( S \) is a subset of \( \mathbb{N} \), and
2. 1 is an element of \( S \), and
3. \( s \in S \) implies \( s' \in S \),

then \( S = \mathbb{N} \).

**Real numbers**

We have also been using the real numbers, which we think of as representing positions on a number line. We will assume that there is a set \( \mathbb{R} \) whose elements we call real numbers. In order to do arithmetic, we need the following axioms.

FULL DISCLOSURE: I want to say that the axioms below are not completely sufficient to determine the set \( \mathbb{R} \), but they are enough to do the arithmetic and algebra we need in this course. When you take “real analysis” (aka either SM333 or SM331H) you will be more careful with these definitions. For instance, the definitions below won’t be enough to deal with notions like “limit” or “convergence”. 
Ring and Field Axioms.

**Axiom 6.** \( \mathbb{N} \) is a subset of \( \mathbb{R} \).

**Axiom 7** (Closure). If \( a \) and \( b \) are real numbers, then \( a + b \) and \( ab \) (or sometimes written \( a \cdot b \)) are real numbers. If \( a \) and \( b \) are natural numbers, then \( a + b \) and \( ab \) are natural numbers.

**Axiom 8** (Associative Laws). If \( a, b \) and \( c \) are real numbers, then \( (a + b) + c = a + (b + c) \) and \( (ab)c = a(bc) \).

**Axiom 9** (Identity Elements). The natural number 1 has the property that if \( a \) is a real number, then \( a \cdot 1 = 1 \cdot a = a \). There is a real number 0 with the property that if \( a \) is a real number, then \( a + 0 = 0 + a = a \).

**Axiom 10** (Inverses). If \( a \) is a real number, then there is a real number \(-a\) with the property that \( a + (-a) = 0 \). If \( a \) is a real number and \( a \) is not 0, then there is a real number \( \frac{1}{a} \) with the property that \( a \cdot \frac{1}{a} = 1 \).

**Axiom 11** (Commutative Laws). If \( a \) and \( b \) are real numbers, then \( a + b = b + a \) and \( ab = ba \).

**Axiom 12** (Distributive Law). If \( a, b, \) and \( c \) are real numbers, then \( a(b+c) = ab + ac \) and \( (a+b)c = ac + bc \).

**Axiom 13.** If \( a, b, \) and \( c \) are real numbers, and \( a = b \), then \( a + c = b + c \) and \( ac = bc \).

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**Order Axioms.** We also assume that we understand what < and > mean for real numbers. We take the following properties as axioms.

**Axiom 14.** If \( a \) and \( b \) are real numbers, then exactly one of the following statements is true:

1. \( a < b \)
2. \( a = b \)
3. \( a > b \).

**Axiom 15.** If \( x, y, \) and \( z \) are real numbers, and \( x < y \) and \( y < z \), then \( x < z \).

**Axiom 16.** If \( x, y, \) and \( z \) are real numbers, and \( y < z \), then \( x + y < x + z \).

**Axiom 17.** If \( x \) and \( y \) are real numbers and \( x > 0 \) and \( y > 0 \), then \( xy > 0 \).