Name: _______________________________  Class Time: 2 days

Purpose: To introduce one example of “substitution method” for double integrals—namely using polar coordinates in \( \mathbb{R}^2 \).

Procedure: Work on the following problems. Bring your ideas and solutions to class to discuss.

Let \( D \) denote the region in the 1st quadrant of the \( xy \)-plane inside the circle \( x^2 + y^2 = 9 \).

Problem 1. Compute the value of the integral: \( \int \int_D x^2 y \, dA \).

Problem 2. Now pretend you are in a Calculus I course. Use the “substitution method” to compute the following integrals.

1. \( \int \frac{x}{1 + x^2} \, dx \)
2. \( \int_0^{\pi/2} \cos^2 \theta \sin \theta \, d\theta \)

Problem 3. In each of the Calculus I integrals above, you set \( u = \) some function of \( x \). Then you had to compute \( du \). Using the \( du \) means “a little bit of \( u \)” language, write a sentence to interpret an expression of the form \( du = ( \text{function of } x ) \, dx \).

The expression \( du \), is an example of a differential. For multivariable functions, say \( z = f(x, y) \), the differential \( dz \) is defined as follows:

\[
dz = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy.
\]

If \( dx \) and \( dy \) are interpreted as “little changes” in \( x \) and \( y \), then the differential is the best linear approximation to a “little change” in \( z \).

Problem 4. Write the conversion equations from polar to rectangular coordinates \( (r, \theta) \mapsto (x, y) \) below:

\[
x = \\
y = 
\]

Problem 5. Compute the differentials \( dx \) and \( dy \).

For the following problems, you may need to refer to your notes on cross products from the beginning of the term!

Problem 6. Find the area of the parallelograms spanned by \( \mathbf{a} \) and \( \mathbf{b} \) in the following examples:

1. \( \mathbf{a} = \langle 1, 1, -2 \rangle, \mathbf{b} = \langle 3, 2, 0 \rangle \)
2. \( \mathbf{a} = \langle 2, 5 \rangle, \mathbf{b} = \langle -1, 3 \rangle \)
3. \( \mathbf{a} = \langle x, 0 \rangle, \mathbf{b} = \langle 0, y \rangle \)

Hint: for 2D vectors, you may want to add a 0 in the \( k \) direction.
Day 2 Homework

Recall that in double integrals $\int\int_D f(x, y) \, dA$, we replace $dA$ with $dx \, dy$ (or $dy \, dx$ depending on the problem) when we actually want to write down and compute a corresponding iterated integral. We have discussed how $dA$ should be interpreted as “a little bit of area”.

**Problem 7.** $dx$ is a little change in the $x$ direction, and $dy$ is a little change in the $y$ direction, so write:

$$a = (dx, 0, 0) \quad b = (0, dy, 0)$$

Just as in Problem 7, find the area of the parallelogram spanned by $a$ and $b$. The answer should be $dA$.

Now we want to mimic Problem 8 to figure out the value of $dA$ in polar coordinates.

**Problem 8.** Recall from Problem 6 that

$$dx = \quad \quad \quad dr + \quad \quad \quad d\theta$$
$$dy = \quad \quad \quad dr + \quad \quad \quad d\theta$$

The goal of this problem is to convince ourselves that the “$r$-direction” and “$\theta$-direction” behave just like $x$ and $y$ in the sense that they are perpendicular to each other. We will use the picture below to explain this.

For the point $(x, y)$ where the circle and two lines are crossing, label $r$ and $\theta$ on the picture. Also, from this point, indicate along the two lines which one corresponds to the $r$-direction and which one corresponds to the $\theta$-direction.

**Problem 9.** The consequence of Problem 9 is that, at each point, we can think of the $r$-direction and $\theta$-direction as a new $i$-direction and $j$-direction in the plane (in particular, notice that it is still true that $i \times j = k$). From this new point of view, the vectors $a$ and $b$ become (use your fill-in-the-blank answers from Problem 9):

$$a = \langle \quad \quad \quad dr, \quad \quad \quad d\theta, 0 \rangle$$
$$b = \langle \quad \quad \quad dr, \quad \quad \quad d\theta, 0 \rangle$$

Now compute $dA$ (which is the area of the parallelogram spanned by $a$ and $b$) in polar coordinates. We get that (fill in the parantheses)

$$dA = \left( \quad \quad \quad \right) \, dr \, d\theta$$
Problem 10. Redo Problem 1, but use polar coordinates.

Problem 11. Suppose that $\Gamma$ is a circular disk in the $xy$-plane, centered at the origin, with radius $a$. Recall that $\iint_{\Gamma} 1 \, dA$ gives the area of $\Gamma$. Use this to prove that the area of $\Gamma$ is $\pi a^2$ (hint: use polar coordinates).

Problem 12. Find the volume under the graph of the paraboloid $z = 16 - 4x^2 - 4y^2$ and above the $xy$-plane (hint: set up a double integral and use polar coordinates).