More notes on sets and functions

**Proposition 1.** If $X$ and $Y$ are sets, then $X \times Y = Y \times X$.

**Proposition 2.** If $X$, $Y$ and $Z$ are sets, then $X \cap (Y \cup Z) = (X \cup Y) \cap (X \cup Z)$.

**Proposition 3.** If $X$, $Y$ and $Z$ are sets, then $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$.

**Proposition 4** (cf. Test 1, Problem 4.c.). Suppose that $X$ and $Y$ are nonempty sets. Then there is a function from $X$ into $Y$.

**Definition 1.** Suppose that $X$ and $Y$ are sets, and that $f$ is a function from $X$ into $Y$. $X$ is called the **domain** of $f$, and $Y$ is called the **codomain** of $f$. The **range** of $f$ is the set $\{ p : \text{there is an element of } f \text{ whose second coordinate is } p \}$.

**Definition 2.** Suppose that $X$ is a set and $Y$ is a set. The statement that $X$ **commands** $Y$ means that there exists a function $f$ from $X$ into $Y$ so that the range of $f$ is $Y$. Furthermore, we say that such an $f$ is **onto**.

Notice that the definition above says that “$X$ commands $Y$” means that “there exists an onto function from $X$ into $Y$”.

**Proposition 5.** If $X$ is a set, then $X$ commands $X$.

**Proposition 6.** Suppose that $X$ and $Y$ are sets and $Y$ is a subset of $X$. Then $X$ commands $Y$.

**Proposition 7.** Suppose that $X$ and $Y$ are sets, and $Y$ is a subset of $X$, and $X$ is not $Y$. Then it is not the case that $Y$ commands $X$.

**Proposition 8.** If $X$ is a set with no elements, and $Y$ is a set with no elements, then $X = Y$.

**Proposition 9.** If $X$, $Y$, and $Z$ are sets, and $Y$ is a subset of $Z$, then $X \times Y$ is a subset of $X \times Z$.

**Proposition 10.** If $X$, $Y$ and $Z$ are sets, then $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$.

**Proposition 11.** The set $\{ t : t \text{ is a real number and } 1 \leq t < 3 \}$ commands the set $\{ u : u \text{ is a real number and } 5 \leq u < 9 \}$.

**Definition 3.** Suppose that $X$ and $Y$ are sets. The **difference of $X$ and $Y$** is $\{ p : p \text{ is an element of } X \text{ and } p \text{ is not an element of } Y \}$.

**Notation.** $X \setminus Y$ stands for the difference of $X$ and $Y$.

**Proposition 12.** If $X$ and $Y$ are sets, then $X \setminus (X \setminus Y) = Y$.

**Proposition 13.** If $X$, $Y$, and $Z$ are sets, then $X \setminus (Y \cup Z) = (X \setminus Y) \cup (X \setminus Z)$.

**Definition 4.** Suppose that $X$ and $Y$ are sets, and that $f$ is a function from $X$ into $Y$. The statement that $f$ is **one-to-one** means that if $(a, c)$ is an element of $f$ and $(b, c)$ is an element of $f$, then $a = b$.

**Proposition 14.** There is a one-to-one function from $\{ 1, 2 \}$ into $\{ 1, 2, 3 \}$.
Proposition 15. There is a function from \( \{1, 2\} \) into \( \{1, 2, 3\} \) which is not one-to-one.

Proposition 16. The set \( \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x^2 \} \) is a one-to-one function from \( \mathbb{R} \) into \( \mathbb{R} \).

Proposition 17. The set \( \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = 3x + 1 \} \) is a one-to-one function from \( \mathbb{R} \) into \( \mathbb{R} \).

Proposition 18. If \( X \) and \( Y \) are sets, and \( X \) is a subset of \( Y \), then there is a one-to-one function from \( X \) into \( Y \).