Day 2 Homework

**Problem 6.** Compute the integral \( \iiint_E f(x, y, z) \, dV \) when \( f(x, y, z) = y \sqrt{x^2 + z^2} \) and \( E \) is the region bounded by the paraboloid \( y = x^2 + z^2 \) and the plane \( y = 4 \). Which variable did you choose to integrate first? Why?

Now let’s learn spherical coordinates! These are analogous in 3D to polar coordinates in 2D.

**Problem 7.** Using the picture below, use trigonometry to write equations for \( x, y, \) and \( z \) in terms of \( \rho, \phi, \) and \( \theta \) [Hint: it is useful to write \( r = |\vec{Q}| \) and use polar coordinates in the \( xy \)-plane as an intermediate step].

![Diagram of spherical coordinates]

\[
x = \\
y = \\
z = 
\]

When we changed double integrals into polar coordinates, we had to make the substitution \( dA \rightarrow r \, dr \, d\theta \) (or \( r \, d\theta \, dr \) depending on the problem). What if we want to do a triple integral in terms of spherical coordinates? Remember that finding the factor of \( r \) boiled down to computing the determinant

\[
\begin{vmatrix}
\frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\
\frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta}
\end{vmatrix}
\]

A similar formula turns out to be true for spherical coordinates.

**Problem 8.** Use your formulas from Problem 7 to compute the relevant partial derivatives, and then compute the \( 3 \times 3 \) determinant

\[
\begin{vmatrix}
\frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\
\frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta}
\end{vmatrix}
\]

Hint: look for opportunities to use \( \cos^2 \alpha + \sin^2 \alpha = 1 \), once with \( \alpha = \phi \) and again with \( \alpha = \theta \).

The result of the problem above is the “stretching factor” for integrals in spherical coordinates. That is, in spherical coordinates

\[
dV \longmapsto (\text{answer to Problem 8}) \, d\rho d\phi d\theta.
\]

**Problem 9.** Compute the integral \( \iiint_E z \, dV \) where \( E \) is the region above the cone \( z = \sqrt{x^2 + y^2} \) and below the sphere \( x^2 + y^2 + z^2 = 16 \). [Hint: use spherical coordinates]