Knowledge Demonstration Opportunity 2: SM221, Calculus III

Name: ALLMAN

28 February 2018

Read all of the following information before starting:

- You are allowed pencils, pens, your TI-36X calculator, and your wits. That is all. In particular, no computers, notes, books, smartphones, iPads, or pocket-sized hobbits.

- To receive full credit, justify your work clearly and in order. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your “final” answer is correct).

- Use sentences to explain your reasoning. Please keep written answers brief; and simultaneously clear!

- Box or otherwise indicate your final numeric answers.

- Good luck!

<table>
<thead>
<tr>
<th>Problem</th>
<th>Possible</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>40</td>
<td></td>
</tr>
</tbody>
</table>
Problem 1 (10 points). Let \( f(x, y) = \frac{1}{3}x^3y - \frac{1}{3}x^3 - \frac{1}{2}y + 2021 \).

(a) Find all critical points of \( f \).

(b) For each critical point, use the second derivative test to determine if it corresponds to a local maximum, local minimum, or saddle point.

(a) First, compute
\[
\nabla f = \left< xy - x^3, \frac{1}{2}x^2 - \frac{1}{2} \right>
\]

Critical pts are places where \( f_x = 0 \) and \( f_y = 0 \). So solve equations:

(A) \( xy - x^3 = 0 \)
(B) \( \frac{1}{2}x^2 - \frac{1}{2} = 0 \)

Eqn (B) factors as \( \frac{1}{2}(x-1)(x+1) = 0 \), so there are two cases:

\( I \) \( x = 1 \)
\( I I \) \( x = -1 \).

In case \( I \), eqn (A) implies \( y - 1 = 0 \) \( \Rightarrow \) \( y = 1 \)

\[ \text{So get Crt. Pt. } (1, 1) \]

In case \( I I \), eqn (A) says \( -y - (-1)^3 = 0 \)

\[ \Rightarrow -y + 1 = 0 \Rightarrow y = 1 \]

\[ \text{So get Crt. Pt. } (-1, 1) \]

(b) Need to know \( f_{xx} = y - 3x^2 \), \( f_{yy} = 0 \)

\[ f_{xy} = f_{yx} = x \]

Thus \( \Delta = f_{xx}f_{yy} - f_{xy}^2 = -x^2 \).

This is always negative (when \( x \neq 0 \)) so for both critical points \( \Delta(1, 1) \) & \( \Delta(-1, 1) \) are negative.

\[ \Rightarrow \text{Both (1,1) & (-1,1) are saddles} \]
Problem 2 (10 points). Answer each of the following multiple choice questions.

(a) Which of the following integrals computes the area of the unit disk (aka the area inside the unit circle)?

(i) $\int_0^{2\pi} \int_0^1 r \, dr \, d\theta$; $\checkmark$

(ii) $\int_0^{2\pi} \int_0^1 r \, dr \, d\theta$; $\checkmark$

(iii) $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \, dx$; $\checkmark$

(iv) $\int_{-1}^1 \int_{-1}^1 dx \, dy$; $\checkmark$

(v) more than one of these; $\checkmark$

(vi) none of these.

(b) Indicate the contour plot below which represents a function that could have a local maximum at (1,1) and a saddle point at (0,0). [Notice that the sketch window for each drawing below is the square $-2 \leq x \leq 2$, $-2 \leq y \leq 2$.]

Contour lines cross at saddles; the domain of integration is a square.

(c) Let $f(x, y)$ be a continuous function which is positive for every point on $0 \leq x \leq 1$, $0 \leq y \leq 1$. Let

$I_1 = \iint_{D_1} f(x, y) \, dA$  $I_2 = \iint_{D_2} f(x, y) \, dA$  $I_3 = \iint_{D_3} f(x, y) \, dA$

Based on the pictures below, what is a correct ordering for the value of the integrals?

(i) $I_1 < I_2 < I_3$; $\checkmark$

(ii) $I_1 < I_3 < I_2$

(iii) $I_2 < I_1 < I_3$

(iv) $I_2 < I_3 < I_1$

(v) $I_3 < I_1 < I_2$

(vi) $I_3 < I_2 < I_1$

Since $f$ is always positive on these pictures, more region to integrate over means more volume under $f$. 
Problem 3 (10 points). Compute the value of the double integral [Hint: sketch the region over which you’re integrating].

\[
\int_0^3 \int_{\sqrt{y^2}}^{\sqrt{y^2+y^2+1}} \left( \frac{1}{x^2+y^2+1} \right) \, dx \, dy
\]

Picture of region

Use polar coordinates

Since the region is part of a disk, and

- The function has \(x^2+y^2\) in it
  (\(2 \times x^2+y^2 \rightarrow r^2\) in polar)

Then we get integral:

\[
\int_0^{\pi/2} \int_0^3 \frac{1}{r^2+1} \cdot r \, dr \, d\theta = \int_0^{\pi/2} \int_0^3 \frac{r}{r^2+1} \, dr \, d\theta.
\]

Since the integrand is a product:

\[
\left( \frac{r}{r^2+1} \right) \times (1)
\]

only \(r\) only \(\theta\) and...

\(\Rightarrow\) All limits of integration are constants,

we can rewrite this as...

\[
\pi \left[ \int_0^{\pi/2} d\theta \right] \cdot \left[ \int_0^3 \frac{r}{r^2+1} \, dr \right] = \pi \left[ \frac{1}{2} \int_1^{10} \frac{du}{u} \right] = \frac{\pi}{2} \ln(u) \bigg|_1^{10} = \frac{\pi}{2} \ln(10)
\]
Problem 4 (10 points). Compute the value of the triple integral \( \iiint_E \cos(x^2) \, dV \), where \( E \) is bounded by

\[
\begin{align*}
  z &= 0 & y &= 0 & z &= \sqrt{x} & y &= \sqrt{x} & x &= 4.
\end{align*}
\]

The region \( E \) is pictured below.

Notice three things:

1. \( \int \cos(x^2) \, dx \) does not have an elementary antiderivative.

2. (Harder to see... but)
   If you integrate \( x \) first on this region, you will need to write the integral as a sum of two integrals.

3. The region \( E \) is symmetric in \( z \) as well.

Together, these facts mean...

- \( \#2 \) \( \Rightarrow \) we don't want to integrate \( x \) first.
- \( \#3 \) \( \Rightarrow \) Even if we wanted to integrate \( x \) first... we can't.
- \( \#1 \) \( \Rightarrow \) Can either integrate \( z \) or \( y \) first, and the order shouldn't matter.

In the end, we get:

\[
\iiint_{E} \cos(x^2) \, dy \, dz \, dx
\]

\( \text{OR} \)

\[
\int_{0}^{4} \int_{0}^{\sqrt{x}} \int_{0}^{\sqrt{x}} \cos(x^2) \, dz \, dy \, dx.
\]
I'll do the first one...

\[ \int_0^4 \int_0^{\sqrt{x}} \int_0^{\sqrt{x}} \cos(x^2) \, dy \, dz \, dx \]

\[ = \int_0^4 \int_0^{\sqrt{x}} \left( y \cos(x^2) \bigg|_{y=0}^{y=\sqrt{x}} \right) \, dz \, dx \]

\[ = \int_0^4 \int_0^{\sqrt{x}} z \sqrt{x} \cos(x^2) \, dz \, dx \]

\[ = \int_0^4 \left. 2z \sqrt{x} \cos(x^2) \right|_{z=0}^{z=\sqrt{x}} \, dx \]

\[ = \int_0^4 x \cos(x^2) \, dx \]

Substitute \( u = x^2 \Rightarrow \frac{1}{2} du = x \, dx \)

\( x=0 \Rightarrow u=0 \)

\( x=4 \Rightarrow u=16 \).

\[ = \frac{1}{2} \int_0^{16} \cos(u) \, du = \frac{1}{2} \sin(u) \bigg|_0^{16} = \left[ \frac{1}{2} \sin(16) \right] \]