Knowledge Demonstration Opportunity 3: SM221, Calculus III

Name: ________________

29 October 2018

Read all of the following information before starting:

- You are allowed pencils, pens, your TI-36X calculator, and your wits. That is all. In particular, no computers, notes, books, smartphones, iPads, or pocket-sized hobbits.

- To receive full credit, justify your work clearly and in order. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your "final" answer is correct).

- Use sentences to explain your reasoning. Please keep written answers brief; and simultaneously clear!

- Box or otherwise indicate your final numeric answers.

- Good luck!

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Problem 1 (10 points). Consider the function \( f(x,y,z) = 12xyz \). Compute the value of the triple integral \( \iiint_E f(x,y,z) \, dV \) where \( E \) is the region below the plane \( z = y \), and above the rectangle in the \( xy \)-plane defined by \( 0 \leq x \leq 2 \) and \( 0 \leq y \leq 3 \).

\[
\int_0^2 \int_0^3 \int_0^y f \, dz \, dy \, dx = \int_0^3 \int_0^2 \int_0^y f \, dz \, dx \, dy
\]

\[
\int_0^3 \int_0^2 \int_0^z f \, dx \, dz \, dy = \int_0^3 \int_0^2 \int_0^z f \, dx \, dy \, dz
\]

\[
\int_0^3 \int_0^2 \int_0^3 f \, dy \, dx \, dz = \int_0^3 \int_0^2 \int_0^z f \, dy \, dz \, dx
\]

First line is easiest (probably).

\[
\int_0^2 \int_0^3 \int_0^y 12xyz \, dz \, dy \, dx = \int_0^2 \int_0^3 6xy^3 \, dy \, dx
\]

\[
= \left[ \int_0^2 6x \, dx \right] \left[ \int_0^3 y^3 \, dy \right] = \left[ 3 \cdot 2^2 \right] \left[ \frac{1}{4} \cdot 3^4 \right]
\]

\[
= 3^5 = 243
\]
Problem 2 (10 points). Answer the following multiple choice questions.

(a) Let \( D \) denote the disk in the \( xy \)-plane bounded by the circle \( x^2 + y^2 = 9 \). If \( D \) is a lamina with density \( \delta(x, y) = x^2 \), which of the following integrals computes the mass of \( D \)?

\[
I_1 = 2 \int_{-3}^{3} \int_{0}^{\sqrt{9-y^2}} x^2 \, dx \, dy,
I_2 = 4 \int_{0}^{3} \int_{0}^{\sqrt{9-y^2}} x^2 \, dx \, dy,
I_3 = \int_{0}^{3} \int_{0}^{\frac{\pi}{2}} r^3 \cos^2(\theta) \, d\theta \, dr.
\]

(i) \( I_1 \) only;
(ii) \( I_2 \) only;
(iii) \( I_3 \) only;
(iv) all of \( I_1, I_2, \) and \( I_3 \);
(v) none of these.

(b) Consider the set \( C \) in 3D defined by the following equations in spherical coordinates:

Which of the following best describes \( C \):

(i) A circle of radius 5;
(ii) A longitude on a sphere of radius 5;
(iii) A latitude on a sphere of radius 5;
(iv) A circle of radius \( \sqrt{5} \).

\[
p = 5 \quad \text{and} \quad \phi = \frac{\pi}{6}
\]

(c) Circle all of the following integrals which evaluate to the number 1.

\[
\begin{align*}
& (i) \int_{0}^{1} \int_{0}^{2\pi} r^2 \, dr \, d\theta \\
& (ii) \int_{0}^{1} \int_{0}^{\pi/2} r^2 \cos^2 \theta \, dr \, d\theta \\
& (iii) \int_{0}^{1} \int_{0}^{\pi/2} \sin \theta \, dr \, d\theta \\
& (iv) \int_{0}^{1} \int_{0}^{\pi/2} \sin \theta \, dr \, d\theta
\end{align*}
\]

Could compute all of these...

or use geometry!

\[
\text{Could compute all of these...}
\]

\[
\text{or use geometry!}
\]

\[
\text{227 latitude}
\]

(longitudes all have same radius.)
Problem 3 (10 points). Compute the value of the iterated integral

\[ \int_0^2 \int_x^{\sqrt{4-x^2}} x \, dy \, dx \]

Region:

\[ y = x \]

\[ y = \sqrt{4-x^2} \]

Notice: \( y = x \) & \( y = \sqrt{4-x^2} \) intersect when:

\[ x = \sqrt{4-x^2} \quad \Rightarrow \quad x^2 = 4 - x^2 \]

\[ 2x^2 = 4 \]

\[ x^2 = 2 \quad \Rightarrow \quad x = \sqrt{2} \]

In polar coordinates, region becomes:

\[ 0 \leq r \leq 2, \quad \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2} \]

Hence, the integral becomes:

\[ \int_0^2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{r^2 \cos^2 \theta}{2} \, r \, d\theta \, dr \]

\[ = \frac{\pi}{3} \left( 1 - \frac{\sqrt{2}}{2} \right) \]

\[ = \left[ \int_0^2 r^2 \, dr \right] \left[ \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos \theta \, d\theta \right] = \left( \frac{1}{3} \cdot 8 \right) \left( 1 - \sin \left( \frac{\pi}{2} \right) - \sin \left( \frac{\pi}{4} \right) \right) \]
We could do Part 3 directly...

\[ \int_0^{\sqrt{2}} \int_0^{\sqrt{4-x^2}} x \, dy \, dx \]

\[ = \int_0^{\sqrt{2}} xy \bigg|_{y=x} \, dx \]

\[ = \int_0^{\sqrt{2}} (x\sqrt{4-x^2} - x^2) \, dx \]

Now we need two terms:

\[ = \int_0^{\sqrt{2}} x\sqrt{4-x^2} \, dx - \int_0^{\sqrt{2}} x^2 \, dx \]

\[ \begin{align*}
  u &= 4-x^2 \\
  du &= -2x \, dx
\end{align*} \]

\[ = \int_0^{\sqrt{2}} -\frac{1}{2} u^{3/2} \, du + \frac{1}{2} \int_0^{\sqrt{2}} u^{1/2} \, du \]

\[ = \frac{1}{3} u^{3/2} \bigg|_{u=4}^{u=2} = \frac{8}{3} - \frac{2}{3} \sqrt{2} \]

Whole thing = \( \frac{8}{3} - \frac{2}{3} \sqrt{2} - \frac{2}{3} \sqrt{2} = \frac{8}{3} (1 - \sqrt{2}) \).
Problem 4 (10 points). Consider the ice-cream-cone-shaped region in 3D bounded below by \( z = \sqrt{x^2 + y^2} \) and bounded above by \( z = \sqrt{2 - x^2 - y^2} \). Let \( I \) denote the half of this ice-cream-cone with \( y \geq 0 \). Compute the volume of \( I \).

\[
\text{Vol} = \iiint_I dV
\]

\[
= \int_0^{\pi/4} \int_0^{\sqrt{2}} \int_0^\pi \rho^2 \sin \phi \ d\rho \ d\phi \ d\theta
\]

\[\text{of course, you could also do volume of whole ice-cream cone (i.e. } \omega \text{)} \& \text{ then just take } \sqrt{2} \text{ by symmetry ...}
\]

\[
\text{Vol} (I) = \left[ \int_0^\pi d\theta \right] \left[ \int_0^{\pi/4} \sin \phi \ d\phi \right] \left[ \int_0^{\sqrt{2}} \rho^2 \ d\rho \right]
\]

\[
= \left[ \pi \right] \left[ -\cos \phi \right]_0^{\pi/4} \left[ \frac{1}{3} \rho^3 \right]_0^{\sqrt{2}}
\]

\[
= \pi \left[ 1 - \frac{\sqrt{2}}{2} \right] \left[ \frac{2}{3} \sqrt{2} \right]
\]

\[\text{OK...}
\]
Problem 4 (10 points). Consider the ice-cream-cone-shaped region in 3D bounded below by \( z = \sqrt{x^2 + y^2} \) and bounded above by \( z = \sqrt{2 - x^2 - y^2} \). Let \( I \) denote the half of this ice-cream-cone with \( y \geq 0 \). Compute the volume of \( I \).

Can use cylindrical coordinates:

\[
\begin{align*}
2 &= \sqrt{2 - x^2 - y^2} \\
\Rightarrow z &= \sqrt{2 - r^2} \\
\end{align*}
\]

\[
\begin{align*}
z &= \sqrt{x^2 + y^2} \\
\Rightarrow z &= r.
\end{align*}
\]

\[
\iiint_I 1 \, dV = \iiint_P \left[ \int_0^{\sqrt{2-r^2}} 1 \, dz \right] \, dA
\]

where \( P \) is in \( xy \)-plane:

\[
x^2 + y^2 = 1
\]

\[
\begin{align*}
\mathbf{r} &= \sqrt{2 - r^2} \\
\Rightarrow r &= \sqrt{2 - r^2} \\
\Rightarrow r &= 1
\end{align*}
\]

\[
\begin{align*}
\frac{\pi}{2} & \int_0^1 \left( \sqrt{2 - r^2} - r^2 \right) \, dr \, d\theta \\
\end{align*}
\]

\[
\begin{align*}
\text{need} \\
u &= 2 - r^2 \\
du &= -2r \\
\end{align*}
\]