Knowledge Demonstration Opportunity 4: SM221, Calculus III

Name: ________________________

6 December 2018

Read all of the following information before starting:

- You are allowed pencils, pens, your TI-36X calculator, and your wits. That is all. In particular, no computers, notes, books, smartphones, iPads, or pocket-sized hobbits.

- To receive full credit, justify your work clearly and in order. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your “final” answer is correct).

- Use sentences to explain your reasoning. Please keep written answers brief; and simultaneously clear!

- Box or otherwise indicate your final numeric answers.

- Good luck!

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Problem 1 (10 points). Throughout this problem, let $\vec{F} = (6xy + 5z, 3x^2 + 2, 5x)$.

(a) Compute curl($\vec{F}$) and div($\vec{F}$).

(b) Compute the value the integral $\int_C \vec{F} \cdot d\vec{r}$ where $C$ is the curve parameterized by

\[ \vec{r} = (t^3 - 2t + 1, t^2, 1 + t), \quad 0 \leq t \leq 1. \]
Problem 2 (10 points). Answer the following multiple choice questions.

(a) Let \( \vec{F} = (\ln(x^4 + 15) + 4x^2y, \sin(y) - 4xy^2) \), and suppose that \( C \) is the curve parameterized by \( \vec{r}(t) = (3\cos(t), 3\sin(t)), \ 0 \leq t \leq 2\pi. \)

Compute the value of the integral \( \int_C \vec{F} \cdot d\vec{r} \).

(i) \(-162\pi\);
(ii) \(-81\pi\);
(iii) 0;
(iv) 81\pi;
(v) 162\pi.

(b) Circle all of the following which parameterize the paraboloid \( x = y^2 + z^2 \)

(i) \( \vec{r}(u, v) = (u^2, u \cos v, u \sin v) \)
(ii) \( \vec{f}(u, v) = (u^2 + v^2, u \cos v, u \sin v) \)
(iii) \( \vec{r}(u, v) = (u^2, u, v) \)
(iv) \( \vec{f}(u, v) = (u^2 + v^2, u, v) \)

(c) Suppose that Hansel and Gretel fly a space-candy retrieving ship along a straight line segment from the point \( A = (0, 1, 0) \) to the point \( B = (2, 2, 2) \). In this problem, all distance is measured in parsecs. The witch, who lives in a space station located at point \( B \), has placed delicious space candy along the well-known space travel route from \( A \) to \( B \) with density \( \delta = x + z \) (in metric tons per cubic parsec). Compute the mass of candy retrieved by Hansel and Gretel on their route from \( A \) to \( B \).

(i) 2 metric tons;
(ii) 3 metric tons;
(iii) 6 metric tons;
(iv) 10 metric tons;
(v) 13 metric tons.
Problem 3 (10 points). Compute the value of the integral \( \iint_S \vec{F} \cdot d\vec{S} \), where

\[ \vec{F} = \langle 3 \cos(y), xy, xz \rangle \]

through the surface \( S \) which bounds the solid wedge-shaped region (depicted at the right) with walls on the planes

\[ x = 0, \quad y = 0, \quad z = 0, \quad z = 2, \quad \text{and} \quad y = 2 - 2x. \]
Problem 4 (10 points). Compute the value of the integral $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$ where

$$\vec{F} = \langle x, x + y, x^3 y^2 z - e^z \rangle$$

and $S$ is the part of the paraboloid $z = 4 - x^2 - y^2$ with $z \geq 0$, oriented so that everywhere its normal vector has positive z-component.
**Formulas**

**Integration Theorems**

**Fundamental Theorem of Line Integrals.** Suppose that $C$ is a curve with parameterization $\vec{r}(t)$ with $a \leq t \leq b$. Suppose that $\vec{F}$ is a vector field and $f$ is a scalar function such that $\nabla f = \vec{F}$. Let $A = \vec{r}(a)$ and $B = \vec{r}(b)$. Then

$$f(B) - f(A) = \int_{C} \vec{F} \cdot d\vec{r}.$$

**Green’s Theorem.** Suppose that $\vec{F} = \langle P, Q \rangle$ is a differentiable vector field on the closed, bounded region $D$ in the plane. Let $C$ denote the positively oriented boundary of $D$. Then,

$$\int_{C} \vec{F} \cdot d\vec{r} = \iint_{D} (Q_{x} - P_{y}) \, dA.$$

**Stokes’s Theorem.** Suppose that $\vec{F}$ is a differentiable vector field defined on the surface $S$ in $\mathbb{R}^{3}$. Let $C$ denote the boundary of $S$, oriented positively with respect to the normal vector on $S$. Then,

$$\int_{C} \vec{F} \cdot d\vec{r} = \iint_{D} \text{curl}(\vec{F}) \cdot d\vec{S}.$$

**Divergence Theorem.** Suppose that $\vec{F}$ is a differentiable vector field defined on the solid region $E$ in $\mathbb{R}^{3}$. Let $S$ denote the surface which is the boundary of $E$, with normal vector oriented outward. Then,

$$\iiint_{S} \vec{F} \cdot d\vec{S} = \iiint_{E} \text{div}(\vec{F}) \, dV.$$

**Alternate coordinate systems**

Polar in 2D.

$$x = r \cos \theta \quad y = r \sin \theta$$

$$x^{2} + y^{2} = r^{2} \quad dA = r \, dr \, d\theta$$

Spherical coordinates in 3D.

$$x = \rho \sin \phi \cos \theta \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi$$

$$\rho^{2} = x^{2} + y^{2} + z^{2} \quad dV = \rho^{2} \sin \phi \, d\rho \, d\phi \, d\theta$$

**Half-angle trig. identities**

For any real number $b$, we have

$$\cos^{2}(b) = \frac{1}{2} (1 + \cos(2b)) \quad \sin^{2}(b) = \frac{1}{2} (1 - \cos(2b)).$$