Knowledge Demonstration Opportunity 4: SM221, Calculus III

Name: _________________________

6 December 2018

Read all of the following information before starting:

- You are allowed pencils, pens, your TI-36X calculator, and your wits. That is all. In particular, no computers, notes, books, smartphones, iPads, or pocket-sized hobbits.
- To receive full credit, justify your work clearly and in order. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your "final" answer is correct).
- Use sentences to explain your reasoning. Please keep written answers brief; and simultaneously clear!
- Box or otherwise indicate your final numeric answers.
- Good luck!

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Problem 1 (10 points). Throughout this problem, let \( \vec{F} = (6xy + 5z, 3x^2 + 2, 5z) \).

(a) Compute \( \text{curl}(\vec{F}) \) and \( \text{div}(\vec{F}) \).

\[
\text{div}(\vec{F}) = \left( \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \right) = \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
6y & 0 & 0 \\
0 & 6x & 0 \\
0 & 0 & 6z
\end{vmatrix} = 6y
\]

\[
\text{curl}(\vec{F}) = \begin{vmatrix}
i & j & k \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
p & q & r
\end{vmatrix} = \begin{pmatrix} 0-0, & 5-5, & 6x-6x \end{pmatrix} = \mathbf{0}
\]

(b) Compute the value of the integral \( \int_C \vec{F} \cdot d\vec{r} \) where \( C \) is the curve parameterized by

\[
\vec{r} = (t^3 - 2t + 1, t^2, 1 + t), \quad 0 \leq t \leq 1.
\]

\[
\text{curl}(\vec{F}) = 0 \implies \vec{F} \text{ is conservative! So find potential } f.
\]

\[
f_x = 6xy + 5z \quad \implies \quad f = \begin{cases} 3x^2y + 5x^2 + c_1(y, z) \end{cases}
\]

\[
f_y = 3x^2 + 2 \quad \implies \quad f = \begin{cases} 3x^2y + 2y + c_2(x, z) \end{cases}
\]

\[
f_z = 5x \quad \implies \quad f = \begin{cases} 5xz + c_3(x, y) \end{cases}
\]

Now -- need "start" & "end" of \( C \) ...

\[
\text{Start: } \vec{r}(0) = (1, 0, 1) \quad \implies \quad \int_C \vec{F} \cdot d\vec{r} = f(0, 1, 2) - f(1, 0, 1)
\]

\[
\]
Problem 2 (10 points). Answer the following multiple choice questions.

(a) Let \( \vec{F} = (\ln(x^4 + 15) + 4x^2y, \sin(y) - 4xy^2) \), and suppose that \( C \) is the curve parameterized by 
\[
\vec{r}(t) = (3 \cos(t), 3 \sin(t)), \quad 0 \leq t \leq 2\pi.
\]
Compute the value of the integral \( \int_C \vec{F} \cdot d\vec{r} \).

(i) \(-162\pi\);
(ii) \(-81\pi\);
(iii) 0;
(iv) \(81\pi\);
(v) \(162\pi\).

\[
\int_C \vec{F} \cdot d\vec{r} = \int \int_D \left( -4y^2 - 4x^2 \right) dA = -4 \int_0^{2\pi} \int_0^{3} r^2 \cdot r \, dr \, d\theta
\]
Use Green's Theorem: \((Q_x - P_y)\)

(b) Circle all of the following which parameterize the paraboloid \( z = y^2 + z^2 \)

(i) \( \vec{r}(u, v) = (u, u \cos v, u \sin v) \)
(ii) \( \vec{r}(u, v) = (u^2 + v^2, u \cos v, u \sin v) \)
(iii) \( \vec{r}(u, v) = (u^2, u, v) \)
(iv) \( \vec{r}(u, v) = (u^2 + v^2, u, v) \)

(c) Suppose that Hansel and Gretel fly a space-candy retrieving ship along a straight line segment from the point \( A = (0, 1, 0) \) to the point \( B = (2, 2, 2) \). In this problem, all distance is measured in parsecs. The witch, who lives in a space station located at point \( B \), has placed delicious space candy along the well-known space travel route from \( A \) to \( B \) with density \( \delta = x + z \) (in metric tons per cubic parsec). Compute the mass of candy retrieved by Hansel and Gretel on their route from \( A \) to \( B \).

(i) 2 metric tons;
(ii) 8 metric tons;
(iii) 6 metric tons;
(iv) 10 metric tons;
(v) 13 metric tons.

\[
\vec{A}_B = <2, 1, 2> \quad \text{param line segment:} \quad \vec{r}(t) = <2t, 1+t, 2t> = A + t \cdot \vec{A}_B
\]
\[
\delta(t, t) = 2t + 2t \quad ds = \sqrt{2^2 + 1^2 + 2^2} \quad ds = 3 \, dt
\]
\[
\int_0^1 (2t + 2t) \cdot 3 \, dt = \int_0^1 6 \, dt = \left[ 6t \right]_0^1 = 6
\]
Problem 3 (10 points). Compute the value of the integral \( \iint_S \vec{F} \cdot d\vec{S} \), where

\[
\vec{F} = (3 \cos(y), xy, xz)
\]

through the surface \( S \) which bounds the solid wedge-shaped region (depicted at the right) with walls on the planes

\[ x = 0, \quad y = 0, \quad z = 0, \quad z = 2, \quad \text{and} \quad y = 2 - 2x. \]

\( \oint \) closed \( \Rightarrow \) use divergence theorem.

\[
\iiint_{\text{Wedge}} \left( 0 + x + x \right) \, dV
\]

\[
= \int_0^1 \int_0^{2-2x} \int_0^2 2x \, dz \, dy \, dx
\]

\[
= \int_0^1 \int_0^{2-2x} 4x \, dy \, dx
\]

\[
= \int_0^1 4x (2 - 2x) \, dx = 8 \int_0^1 x (1 - x) \, dx
\]

\[
= 8 \left[ \frac{1}{2} - \frac{1}{3} \right] = 8 \cdot \frac{1}{6} = \boxed{\frac{4}{3}}
\]
Problem 4 (10 points). Compute the value of the integral \( \iint_S \text{curl}(\vec{F}) \cdot d\vec{S} \) where \( \vec{F} = (x, x+y, x^2y^2z - e^z) \)

and \( S \) is the part of the paraboloid \( z = 4 - x^2 - y^2 \) with \( z \geq 0 \), oriented so that everywhere its normal vector has positive \( z \)-component.

\[ \int \int_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}. \]

\( \vec{C} \) is param. by \( \vec{r} = \langle 2\cos t, 2\sin t, 0 \rangle \) \( \alpha t \leq 2\pi \)

\[ d\vec{r} = \langle -2\sin t, 2\cos t, 0 \rangle dt. \]

So \[ \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle 2\cos t, 2\cos t + 2\sin t, \text{stuff} \rangle \]

\[ \langle -2\sin t, 2\cos t, 0 \rangle dt \]

\[ = \int_0^{2\pi} (-4 \cos t \sin t + 4\cos^2 t + 4\cos t \sin t) dt \]

\[ = \int_0^{2\pi} 4\cos^2 t dt = \int_0^{2\pi} (1 + \cos 2t) dt \]

\[ H\text{alf-angle formula.} \]

\[ = 4\pi. \]
Problem 4 (10 points). Compute the value of the integral \( \iint_S \text{curl}(\vec{F}) \cdot d\vec{S} \) where
\[
\vec{F} = (x, x + y, x^2 y^2 z - e^z)
\]
and \( S \) is the part of the paraboloid \( z = 4 - x^2 - y^2 \) with \( z \geq 0 \), oriented so that everywhere its normal vector has positive \( z \)-component.

\[
\text{Ok, you can do the problem as follows...}
\]

Here's \( S \):

\[
\text{which is NOT closed.}
\]

But we can add the disk on the bottom with downward orientation to close it.

Then \( \Phi_{ST} + \Phi_{DV} = \iiint_\text{solid \ paraboloid} \text{div(curl}(\vec{F})) \, dV \)

\[
\text{div(curl} \vec{F}) = 0 \quad \text{always}
\]

\[
\Rightarrow \Phi_{ST} = \Phi_{DV}.
\]

Now on \( D \) (w/ up. orientation...), \( d\vec{S} = \vec{r} \, dA \)
\[
\text{Thus:} \quad \Phi_{DV} = \iiint_D \vec{r} \cdot d\vec{A} = \iint_D \left< 0, 0, 1 \right> \, dA
\]
\[
= \iint_D (1 - 0) \, dA = \text{Area} \left( D \right) = 4\pi.
\]