Knowledge Demonstration Opportunity 2: SM221P, Calculus III

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Read all of the following information before starting:

- You are allowed pencils, pens, your TI-36X calculator, and your wits. That is all. In particular, no computers, notes, books, smartphones, iPads, or pocket-sized hobbits.

- To receive full credit, justify your work clearly and in order. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your “final” answer is correct).

- Use sentences to explain your reasoning. Please keep written answers brief; and simultaneously clear!

- Box or otherwise indicate your final numeric answers.

- Good luck!

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<th>Problem</th>
<th>Possible</th>
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Problem 1 (10 points). Let $f(x, y) = \ln (9 - x^2 - y^2)$.

(a) Describe (with correct inequalities, or a drawing, or words) the domain and range of $f$.

Need $9 - x^2 - y^2 > 0$  
(since $\ln(-)$ can't take negative numbers)

Range: $\lim_{z \to 0} \ln(z) = -\infty$, so $0 < z \leq \ln(9)$

Words: A disk of radius 3, centered on the origin, in the $xy$-plane, not including the boundary circle $x^2 + y^2 = 9$

(b) Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

\[ f_x = \frac{-2x}{9 - x^2 - y^2} \]

\[ f_y = \frac{-2y}{9 - x^2 - y^2} \]

(c) Compute the value of the directional derivative $D_v f(2, 1)$ with $v$ parallel to $(3, -4)$.

Need $D_v f(2, 1) = \nabla f(2, 1) \cdot \frac{v}{|v|}$

\[ = \left\langle \frac{-4}{9 - 4 - 1}, \frac{-2}{9 - 4 - 1} \right\rangle \cdot \frac{\langle 3, -4 \rangle}{\sqrt{3^2 + 4^2}} \]

\[ = \left\langle -1, -\frac{1}{2} \right\rangle \cdot \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle = -\frac{3}{5} + \frac{2}{5} = -\frac{1}{5} \]
Problem 2 (10 points). Each of the pictures below is a contour plot for some function \( f(x, y) \). Use the plots to answer the questions below. Circle your final answer.

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<tr>
<td><img src="image1" alt="Contour Plot I" /></td>
<td><img src="image2" alt="Contour Plot II" /></td>
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<tr>
<td><img src="image3" alt="Contour Plot III" /></td>
<td><img src="image4" alt="Contour Plot IV" /></td>
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(a) In contour plot III, suppose that the positive \( y \)-direction is north and the positive \( x \)-direction is east. Which of the following most accurately describes the direction of the gradient \( \nabla f(0, 0) \)?

(i) north  
(ii) south  
(iii) east  
(iv) west

(b) For which of the contour plots is the partial derivative \( f_y(1, 1) \) positive?

(i) I and II  
(ii) III and IV  
(iii) all of these  
(iv) none of these

(c) Which of the following numbers is larger?

(i) \( f_{yy}(0, -1.5) < 0 \) on IV

(ii) \( f_{xx}(-1, 1) > 0 \) on III

(iii) neither, they are roughly equal
Problem 3 (10 points). At time $t = 0$, a particle is at the point $(0, -1, 0)$ and has an initial velocity of $(1, 0, 1)$. Moreover, for all $t \geq 0$, the particle experiences an acceleration

$$a(t) = \left\langle - \sin(t), \cos(t), \frac{1}{2}(t + 1)^{-1/2} \right\rangle.$$

(a) Compute parameterizations for the velocity and position vectors of the particle (i.e. find vectors $v(t)$ and $r(t)$).

$$\vec{v}(t) = \left\langle \cos t, \sin t, \frac{(t+1)^{3/2}}{2} \right\rangle + \left\langle c_1, c_2, c_3 \right\rangle$$

$$\vec{v}(0) = \left\langle 1, 0, 1 \right\rangle + \left\langle c_1, c_2, c_3 \right\rangle = \left\langle 1, 0, 1 \right\rangle$$

$$\Rightarrow c_1 = c_2 = c_3 = 0$$

$$\Rightarrow \vec{v}(t) = \left\langle \cos t, \sin t, \frac{(t+1)^{3/2}}{2} \right\rangle$$

Now:

$$\vec{r}(t) = \left\langle \sin t, - \cos t, \frac{2}{3}(t+1)^{3/2} \right\rangle + \left\langle d_1, d_2, d_3 \right\rangle$$

$$\vec{r}(0) = \left\langle 0, -1, \frac{2}{3} \right\rangle + \left\langle d_1, d_2, d_3 \right\rangle = \left\langle 0, -1, 0 \right\rangle$$

$$\Rightarrow d_1 = d_2 = 0 \quad j \quad d_3 = -\frac{2}{3}$$

$$\Rightarrow \vec{r}(t) = \left\langle \sin t, - \cos t, \frac{2}{3}(t+1)^{3/2} - \frac{2}{3} \right\rangle$$

(b) Use your answer from (a) to compute the arclength of the particle's path from $t = 2$ to $t = 7$.

Need to know speed $v(t) = \left\| \vec{v}(t) \right\| = \sqrt{\cos^2 t + \sin^2 t + (t+1)}$

$$= \sqrt{1 + t + 1}$$

$$= \sqrt{t+2}$$

Now integrate!

$$s = \int_2^7 \sqrt{t+2} \, dt = \frac{2}{3}(t+2)^{3/2} \bigg|_2^7 = \frac{2}{3} \left( 9^{3/2} - 4^{3/2} \right)$$

$$= \frac{2}{3} \left( 27 - 8 \right) = \frac{2}{3} (19) = \frac{38}{3} = 12 + \frac{2}{3}$$
Problem 4 (10 points). For $0 \leq u \leq \pi$ and $0 \leq v \leq \pi/4$, consider the parameterized surface in $\mathbb{R}^3$ given by $$S(u, v) = (3 \cos(u) \sin(v), 3 \sin(u) \sin(v), 3 \cos(v)).$$

(a) Identify the image below which depicts the range of the function $S(u, v)$ [HINT: recall that the spherical coordinates in $\mathbb{R}^3$ satisfy the equations $x = \rho \cos \theta \sin \phi$, $y = \rho \sin \theta \sin \phi$, $z = \rho \cos \phi$].

(b) Suppose that the temperature in the region of space near the surface $S(u, v)$ is given (in $C^\infty$) by the function $T(x, y, z) = \frac{x}{z^2 + 1}$. Use the chain rule to compute the rate of change in temperature in the "$v$ direction" along the surface at the point corresponding to $u = 0$ and $v = 0$.

\[
\begin{align*}
&\text{Want:} \quad \frac{\partial T}{\partial v} \bigg|_{u=0, v=0} \quad \text{Ch. Rule} \quad \frac{\partial T}{\partial v} = \frac{\partial T}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial T}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial T}{\partial z} \frac{\partial z}{\partial v} \\
&\frac{\partial T}{\partial x} = \frac{1}{z^2 + 1}, \quad \frac{\partial T}{\partial y} = 0, \quad \frac{\partial T}{\partial z} = \frac{-2xz}{(z^2 + 1)^2} \\
&\text{Now } S(0, 0) = (0, 0, 3) \quad \text{so} \quad \frac{\partial T}{\partial x} \bigg|_{(0,0,3)} = \frac{1}{10}, \quad \frac{\partial T}{\partial y} \bigg|_{(0,0,3)} = 0 \\
&\text{Thus:} \quad \frac{\partial T}{\partial v} \bigg|_{u=0, v=0} = \frac{\partial T}{\partial x} \bigg|_{(0,0,3)} \cdot \frac{\partial x}{\partial v} \bigg|_{(0,0)} + \frac{\partial T}{\partial y} \bigg|_{(0,0,3)} \cdot \frac{\partial y}{\partial v} \bigg|_{(0,0)} = 0.
\end{align*}
\]