Read all of the following information before starting:

- You are allowed pencils, pens, and your wits. That is all. In particular, no computers, notes, books, smartphones, iPads, pocket-sized hobbits, or outside assistance of any kind.

- To receive full credit, justify your answers clearly and in order. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your “final” answer is correct) or if I cannot understand your written justifications (even if you are able to explain them out loud).

- Use sentences to explain your reasoning. Please keep written answers brief; and simultaneously clear! You may use mathematical shorthands such as the symbols ∀, ∃, ⇒, ⇔, etc. but you must use them correctly!

- Good luck and try to have fun!

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**Problem 1** (10 points). Let $U = \{u \in \mathbb{R} : -3 < u < 3\}$ and $V = \{v \in \mathbb{R} : 2 \leq v \leq 5\}$. Write each of the following sets in set builder notation.

a. $U \setminus V$

b. $U \cap V$

c. $U \cup V$

d. $(U \cup V) \setminus (U \cap V)$

e. $(U \cap V) \setminus (U \cup V)$
Problem 2 (5 points). Choose one (and only one) of the following “famous” theorems, and prove them.

a. There are infinitely many prime numbers.

b. $\sqrt{2} \notin \mathbb{Q}$. 
Problem 3 (16 points). Prove or disprove the following statements.

a. If $X$, $Y$, and $Z$ are sets, then $X \setminus (Y \cap Z) = (X \setminus Y) \cup (X \setminus Z)$.

b. If $X = \{a \in \mathbb{R} : 0 \leq a \leq 5\}$ and $Y = \{b \in \mathbb{R} : 0 \leq b \leq 15\}$, then $X$ commands $Y$.

c. If $X = \{0, 1, 2, 3, 4, 5\}$ and $Y = \{0, 1, 2, 3, \ldots, 15\}$ then $X$ commands $Y$.

d. If $n \in \mathbb{Z}$ and $n^2$ is even, then $n$ is even. [You may assume and freely use, without stating them, any of the axioms regarding arithmetic or order of real numbers. Further, you may use freely that every integer is either even or odd, but cannot be both.]
Problem 4 (8 points). Prove the following formula by induction on $n$;

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$ 

Observe that the first equality is just the definition of the “sigma notation” for sums. You need to prove that the second equality holds.
**Problem 5** (8 points). Let $X$ and $Y$ be non-empty sets, and let $f$ be a function from $X$ into $Y$. For any subset $A \subseteq X$, define the set

$$f(A) = \{ b \in Y : \exists a \in A \text{ such that } (a, b) \in f \}.$$ 

We call $f(A)$ the “image of $A$ under $f$.” Observe that $f(X) = \text{Range}(f)$.

a. Let $X = \{ t, $, @, 2020, 2021 \}$, $Y = \{ \Phi, \Upsilon, \nu, \mu, \alpha, \Theta \}$, $A = \{ 2020, 2021 \}$, and

$$f = \{ (t, \alpha), (\$, \alpha), (@, \Theta), (2020, \Theta), (2021, \Phi) \}.$$ 

Write the set $f(A)$ in roster notation.

b. Let $X = \{ x \in \mathbb{R} : -3\pi \leq x \leq 3\pi \}$, $Y = \mathbb{R}$, $A = \{ a \in \mathbb{R} : -\pi/6 \leq a \leq \pi/6 \}$, and $f$ is the sine function, given as a set by $f = \{ (x, \sin(x)) : x \in X \}$. Write the set $f(A)$ in set builder notation.

c. [4 BONUS POINTS] Let $X$ and $Y$ be any non-empty sets, $f$ any function from $X$ into $Y$, and suppose that both $A_1 \subseteq X$ and $A_2 \subseteq X$ (observe that these hypotheses automatically imply that $A_1 \cap A_2 \subseteq X$). Prove or disprove: $f(A_1) \cap f(A_2) \subseteq f(A_1 \cap A_2)$. 

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