SM212P Section 8.2 Worksheet (Melles)

Solve each system of differential equations eigenvalues and eigenvectors.

Example

$$x' = 4x + y, \quad x(0) = 1$$

$$y' = x + 4y, \quad y(0) = 3$$

Let $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$. The system can be written in **matrix form** as

$$X' = AX \text{ and } X(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

The eigenvalues of A are $\lambda_1 = 5$ and $\lambda_2 = 3$. The corresponding eigenvectors are $K_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $K_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Let

$$X_1 = K_1 e^{\lambda_1 t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} \text{ and } X_2 = K_2 e^{\lambda_2 t} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{3t}.$$

Then $X_1' = \lambda_1 X_1 = AX_1$ and $X_2' = \lambda_2 X_2 = AX_2$, so X_1 and X_2 are solutions of the equation $X' = AX$.

The solutions X_1 and X_2 are **linearly independent** on $(-\infty, \infty)$ since the matrix with columns X_1 and X_2 (the **Wronskian** of X_1 and X_2) is nonzero on $(-\infty, \infty)$:

$$W(X_1, X_2) = \begin{vmatrix} e^{5t} & -e^{3t} \\ e^{5t} & e^{3t} \end{vmatrix} = 2e^{8t}.$$

The general solution of the equation $X' = AX$ is

$$X = c_1 X_1 + c_2 X_2 = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{3t}.$$

We solve for the constants c_1 and c_2 using the condition $X(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and find $c_1 = 2$ and $c_2 = 1$. Therefore,

$$X = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{3t},$$

so

$$x = 2e^{5t} - e^{3t} \text{ and } y = 2e^{5t} + e^{3t}.$$

1.

$$x' = x + 2y, \quad x(0) = 3$$

$$y' = 4x + 3y, \quad y(0) = 0$$

2.

$$x' = \frac{1}{2}x, \quad x(0) = 3$$

$$y' = x - \frac{1}{2}y, \quad y(0) = 5$$

Solutions:

1.

$$x = e^{5t} + 2e^{-t},\ y = 2e^{5t} - 2e^{-t}$$

2.

$$x = 3e^{\frac{1}{2}t},\ y = 3e^{\frac{1}{2}t} + 2e^{-\frac{1}{2}t}$$