Basic feasible solutions: A basic solution which is nonnegative.

Basic solution: For a canonical form linear program (see below), a basic solution is a vector \( x \) where, for a given basis, \( B \), if \( j \notin B \) then \( x_j = 0 \). The variables \( x_j \) for \( j \in B \) solve the square linear system \( Bx(B) = b \). Thus, because \( B \) is invertible, \( x(B) = B^{-1}b \).

Basic variable: For a basic solution, \( x \), with basis \( B \), any variable \( x_j \) where \( j \in B \).

Basis: For a canonical form linear program (see below), a basis is a set, \( B \), of indices corresponding to \( m \) linearly independent columns. For an \( n \) dimensional vector \( x \), we write \( x(B) \) to denote a vector with components \( x_j \) for \( j \in B \). For the matrix \( A \), we write \( A(B) \) to denote the submatrix from taking the \( m \) columns of \( A \) corresponding to \( B \).

Bland’s rule: An anti-cycling pivot rule where the entering variable is the nonbasic variable with a positive reduced cost and the smallest index. If the min. ratio test results in a tie, then the leaving variable is the basic variable chosen is the one with the smallest index.

Blending constraint: For a product made from a “blend” of different items, a constraint such that the percentage of one or more of the items (or even a characteristic of an item) is constrained. To linearize such a constraint, the denominator is typically multiplied through (which assumes that at least a positive amount of the product is made). For example, suppose that a gasoline is made by blending two types of crude oil, type \( A \) and type \( B \). Let \( x_A \) denote the amount of oil \( A \) used and \( x_B \) the amount of oil \( B \) used. If \( A \) has an octane rating of 93 and \( B \) an octane rating of 88, then to keep the average octane rating of the gas at least 91, the nonlinear constraint

\[
\frac{93x_A + 88x_B}{x_A + x_B} \geq 91
\]

could be used. To make this constraint linear and therefore valid for a linear program, use:

\[
93x_A + 88x_B \geq 91(x_A + x_B).
\]

Canonical form linear program: A linear program in the form

\[
\begin{align*}
\text{max or min} & \quad c^\top x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0,
\end{align*}
\]

where \( A \) is an \( m \times n \) real matrix with full-row rank, \( c \) is an \( n \) dimensional real vector, and \( b \) is an \( m \) dimensional vector. The decision variables are the \( n \)-dimensional vector \( x \). Note that the objective can be minimization or maximization.

Combinatorial Optimization Problem: A mathematical program where the variables are restricted to zero or one.

Continuous Variable: A decision variable that can take non-integer values.

Dantzig’s rule: A pivot rule where the entering (nonbasic) variable is the one with the largest reduced cost (for a maximization problem).

Decision Variable: A variable in a mathematical program that can be changed.

Decision Vector: A vector of some or all (usually all) of the decision variables in a mathematical program.

Degenerate basic feasible solution: A basic feasible solution where one or more of the basic variables is zero.

Discrete Variable: A decision variable that can only take integer values.

Feasible Solution: A decision variable that satisfies all the constraints.

Feasible Region: The set of all feasible solutions, i.e., \( S \).
Graph: A set, $\mathcal{V}$, of nodes (or vertices) and a set of pairs of nodes, called edges (or arcs), $\mathcal{E}$, typically written $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and can be drawn as (arbitrarily placed) small circles in space for each $i \in \mathcal{V}$ with a line between nodes $i$ and $j$ if $(i, j) \in \mathcal{E}$. We typically use directed graphs, which means the order of the edge matters, i.e., $(i, j) \neq (j, i)$ for $i \neq j$. Note that we also assume that graphs do not have loops, i.e., edges from a node to itself, or multiedges, multiple copies of an edge.

Infeasible Problem: A mathematical program where $S = \emptyset$, i.e., where no feasible solution exists.

Infeasible Solution: A solution, $\mathbf{x}$, that does not satisfy the constraints, i.e., $\mathbf{x} \notin S$.

Linear Program (LP): A mathematical program where $f$ is linear and $S$ is determined by linear inequalities and equalities. This can be written as

$$\begin{align*}
\text{max} & \quad c_1 x_1 + \ldots + c_n x_n \\
\text{s.t.} & \quad a_{i1} x_1 + \ldots + a_{in} x_n \leq b_i, i = 1, \ldots, m.
\end{align*}$$

We can show that every LP can be written in this form. Linear programs assume additivity, certainty, divisibility, and proportionality.

Mathematical Program: Also called an optimization problem, a mathematical program consists of a given set $S \subset \mathbb{R}^n$ and function $f : S \to \mathbb{R}$. Then, the mathematical program is

$$\max \{ f(\mathbf{x}) : \mathbf{x} \in S \} \text{ or } \min \{ f(\mathbf{x}) : \mathbf{x} \in S \}.$$ 

Minimum ratio test: For a canonical form linear program, basic solution, $\mathbf{x}$, with basis, $\mathcal{B}$, and simplex direction $\mathbf{d}^{(j)}$, the minimum ratio test is the calculation used to determine the maximum step size, $\lambda$, that can be used before $\mathbf{x} + \lambda \mathbf{d}^{(j)}$ is not feasible. Because $\mathbf{x} + \lambda \mathbf{d}^{(j)}$ satisfies equality constraints for any $\lambda$, the step size is determined in order to satisfy the nonnegativity constraints, i.e., $\lambda$ is the maximum value such that, for all $i$,

$$x_i + \lambda d_i^{(j)} \geq 0.$$ 

For nonbasic indices $i$, $x_j + \lambda d_j^{(j)} \geq 0$ so the minimum ratio test is calculated via

$$\lambda = \min \{ \frac{x_i}{-d_i^{(j)}} : i \in \mathcal{B}, d_i < 0 \}.$$ 

Nonbasic variable: For a basic solution, $\mathbf{x}$, with basis $\mathcal{B}$, any variable $x_j$ where $j \notin \mathcal{B}$. All nonbasic variables are zero, i.e., $x_j = 0$ for $j \notin \mathcal{B}$.

Pivot rule: How an entering is picked in Simplex. Pivot rules are only used if there is an improving direction (i.e., at least one positive reduced cost for a max problem or at least one negative reduced cost for a min problem). Some pivot rules also specify how leaving variables are chosen in the event of a tie in the minimum ratio test.

Optimal Solution: A feasible solution, $\mathbf{x}^*$, that has, for all $\mathbf{x} \in S$:

$$f(\mathbf{x}^*) \geq f(\mathbf{x}) \text{ for a maximization problem, or;}$$

$$f(\mathbf{x}^*) \leq f(\mathbf{x}) \text{ for a minimization problem.}$$

Reduced cost: For a canonical form linear program and a basic feasible solution with basis, $\mathcal{B}$, and basis matrix $B = A(\mathcal{B})$, the reduced cost of the nonbasic variable with index $j \notin \mathcal{B}$ is the directional derivative of the objective function, $\mathbf{c}^T \mathbf{x}$, in the simplex direction $\mathbf{d}^{(j)}$, i.e.,

$$\tau_j = D_{\mathbf{d}^{(j)}}(\mathbf{c}^T \mathbf{x}) = \nabla(\mathbf{c}^T \mathbf{x}) \cdot \mathbf{d}^{(j)} = \mathbf{c} \cdot \mathbf{d}^{(j)} = c_j - \mathbf{c}^T B^{-1} A_j.$$ 

Reduced cost optimality conditions: For a canonical form linear program, a basic feasible solution with basis, $\mathcal{B}$, is optimal for a maximization problem if, for all $j \notin \mathcal{B}$,

$$\tau_j \leq 0.$$ 

For a minimization problem, the condition is that for all $j \notin \mathcal{B}$,

$$\tau_j \geq 0.$$
Simplex direction: For a canonical form linear program and a basic feasible solution, \( x \), with basis, \( B \), and basis matrix \( B = A(B) \), the simplex direction of the nonbasic variable with index \( j \not\in B \) is an \( n \)-dimensional vector \( d^{(j)} \) where \( d^{(j)}_j = 1 \), for \( i \not\in B, i \neq j \), \( d^{(j)}_i = 0 \) and

\[
d^{(j)}(B) = -B^{-1}A_j.
\]

Note that \( A_j \) is the \( j \)th column of \( A \). Also recall that the derivation of \( d^{(j)} \) was based on (1) changing only one non-basic variables, and (2) satisfying all the equality constraints for any change from the basic feasible solution, \( x \), i.e., for any \( \lambda \) and \( j \not\in B \).

\[
A(x + \lambda d^{(j)}) = b.
\]

Solution: A particular setting of the decision vector. Note that the solution may not satisfy all the constraints.

Unbounded Mathematical Program: A mathematical program where for any \( K > 0 \), for a maximization, there exists a feasible solution \( x \) with \( f(x) > K \). For a minimization problem, for any \( K < 0 \) there is a feasible solution \( x \) with \( f(x) < K \).