Lesson 32: LP Duality and Game Theory
SA305 – Linear Programming

Spring 2013
Today...

- LP duality and two-player zero-sum game theory
Game theory
Game theory

- Game theory is the mathematical study of strategic interactions, in which an individual’s success depends on his/her own choice as well as the choices of others.

- We’ll look at one type of game, and use LP duality to give us some insight about behavior in these games.
Two-player zero-sum games

- Two players make decisions simultaneously
- Payoff depends on joint decisions
- Zero-sum: whatever one person wins, the other person loses

Examples:
  - Rock-paper-scissors
  - Advertisers competing for market share
    (gains/losses over existing market share)
Payoff matrices

- 2 players
  - player R (for “row”)
  - player C (for “column”)

- Player R chooses among $m$ rows (actions)

- Player C chooses among $n$ columns

- Example: rock-paper-scissors, $m = 3, n = 3$

<table>
<thead>
<tr>
<th></th>
<th>Rock</th>
<th>Paper</th>
<th>Scissors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rock</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>Paper</td>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>Scissors</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

- This is the **payoff matrix** for player R

- Zero-sum: Player C receives the negative
Payoff matrices

- Another example: $m = 2$, $n = 3$

<table>
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<th>3</th>
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<tbody>
<tr>
<td>1</td>
<td>-2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

- Player R chooses row 2, Player C chooses column 1

- What is the payoff of each player?
An experiment

- Two volunteers?
An experiment

- Two volunteers?
- 2 players, 3 strategies
- Payoff matrix:

\[
\begin{array}{ccc}
1 & 2 & 3 \\
1 & -2 & 1 & 2 \\
2 & 2 & -1 & 0 \\
3 & 1 & 0 & -2 \\
\end{array}
\]

- Play this game for 5 trials
Pure and mixed strategies

- **Pure strategy**: pick one row (or column) over and over again

- **Mixed strategy**: each player assigns probabilities to each of his/her strategies

For example:

<table>
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<tbody>
<tr>
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<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>-2</td>
</tr>
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</table>

Suppose player R plays all three actions with equal probability

- Row 1 with probability 1/3
- Row 2 with probability 1/3
- Row 3 with probability 1/3
Pure and mixed strategies

For example:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
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<th>3</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-2</td>
<td>1</td>
<td>2</td>
<td>1/3</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>-1</td>
<td>0</td>
<td>1/3</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>-2</td>
<td>1/3</td>
</tr>
</tbody>
</table>

Expected payoffs

Suppose player R plays all three actions with equal probability.

⇒ Can compute expected payoffs:

- If player C plays
  - column 1:
  - column 2:
  - column 3:
Who has the advantage?

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<td>0</td>
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- Can we find “optimal” (mixed) strategies for two-player zero-sum games?

- What can player R guarantee in return, regardless of what C chooses?
**Player R and payoff lower bounds**

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<td>3</td>
<td>1</td>
<td>0</td>
<td>-2</td>
<td>1/3</td>
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<tr>
<td>Expected payoff</td>
<td>1/3</td>
<td>0</td>
<td>0</td>
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</table>

Suppose Player R plays all three actions with equal probability.

With this mixed strategy, R can guarantee a payoff of at least:

This is a lower bound on the payoff R gets when playing \((1/3, 1/3, 1/3)\)
Player C and payoff upper bounds

<table>
<thead>
<tr>
<th></th>
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<th>3</th>
<th>Expected payoff (for R)</th>
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<tbody>
<tr>
<td>1</td>
<td>-2</td>
<td>1</td>
<td>2</td>
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<td>2</td>
<td>-1</td>
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<td>1</td>
<td>0</td>
<td>-2</td>
<td></td>
</tr>
<tr>
<td>Prob.</td>
<td>1/3</td>
<td>1/3</td>
<td>1/3</td>
<td></td>
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</table>

- Player C’s payoff = −(Player R’s payoff)
- Player C wants to limit Player R’s payoff
- Suppose Player C plays all three actions with equal probability
- With this mixed strategy, C can guarantee that R gets a payoff of at most:
- This is an upper bound on the payoff R gets when C plays (1/3, 1/3, 1/3)
Let’s optimize: Player R’s problem

- Want to decide mixed strategy that maximizes guaranteed payoff

$\Rightarrow$ Decision variables:

$x_i = \text{prob. of choosing action } i$

for $i \in \{1, 2, 3\}$

- Optimization model:
Let’s optimize: Player R’s problem

- Player R’s problem: maximin
Let’s optimize: Player R’s problem

- Player R’s problem: maximin
- Convert Player R’s problem to LP:
Player C’s problem

- Want to decide mixed strategy that limits Player R’s payoff

⇒ Decision variables:

\[ y_i = \text{prob. of choosing action } i \]
for \( i \in \{1, 2, 3\} \)

- Optimization model:

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| Prob. | \( y_1 \) | \( y_2 \) | \( y_3 \) |
Player C’s problem

- Player C’s problem: minimax

- Convert Player C’s problem to LP:
Optimal mixed strategy for Player R

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<tbody>
<tr>
<td>1</td>
<td>-2</td>
<td>1</td>
<td>2</td>
<td>7/18</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>-1</td>
<td>0</td>
<td>5/18</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>-2</td>
<td>1/3</td>
</tr>
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Expected payoff | 1/9 | 1/9 | 1/9 |

- Solve Player R’s LP

⇒ Optimal mixed strategy for R guarantees that R can get at least

\[
\min\{\frac{1}{9}, \frac{1}{9}, \frac{1}{9}\} = \frac{1}{9}
\]

- “Maximin” payoff = 1/9
Optimal mixed strategy for Player C

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<td>1/9</td>
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<td>-1</td>
<td>0</td>
<td>1/9</td>
</tr>
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<td>1</td>
<td>0</td>
<td>-2</td>
<td>1/9</td>
</tr>
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</table>

Prob. | 1/3 | 5/9 | 1/9 |

- Solve Player C’s LP

⇒ Optimal mixed strategy for C guarantees that C can limit R’s payoff to at most

\[
\max\{\frac{1}{9}, \frac{1}{9}, \frac{1}{9}\} = \frac{1}{9}
\]

- “Minimax” payoff = \(1/9\)

- “Maximin” payoff = “Minimax” payoff - not a coincidence
Fundamental Theorem of 2-Player 0-Sum Games

- For any 2-player 0-sum game, let

\[ p_R(x) = \text{lower bound on R's payoff if R plays probability vector } x \]
\[ p_C(y) = \text{upper bound on R's payoff if C plays probability vector } y \]

- Let

\[ x^* = \text{maximizer of } p_R \]
\[ y^* = \text{minimizer of } p_C \]

- Then,

\[ p_R(x^*) = p_C(y^*) \]

- That is, maximin payoff = minimax payoff
Why is this remarkable?

Think back to example

Imagine you are Player R, and you have to announce in advance what your mixed strategy is

Intuitively, this seems like a bad idea

But, if you play the optimal maximin strategy, you are guaranteed an expected payoff of $1/9$

And, Player C cannot do anything to prevent this

Announcing the strategy beforehand does not cost you in this case
Why is this true?

- Turns out Player R’s LP and Player C’s LP is a primal-dual pair
- Proof follows immediately from strong duality for LP
- For example, after some manipulation, it is easy to see that in our game, Player R’s LP and Player C’s LP are duals of each other
Fundamental Theorem of 2-Player 0-Sum Games

Player R’s LP:

\[
\begin{align*}
\text{max} & \quad z \\
\text{s.t.} & \quad 2x_1 - 2x_2 - x_3 + z \leq 0 \\
& \quad -x_1 + x_2 + z \leq 0 \\
& \quad -2x_1 + 2x_3 + z \leq 0 \\
& \quad x_1 + x_2 + x_3 = 1 \\
& \quad x_1, \quad x_2, \quad x_3 \geq 0
\end{align*}
\]

Player C’s LP:

\[
\begin{align*}
\text{min} & \quad w \\
\text{s.t.} & \quad 2v_1 - v_2 - 2v_3 + w \geq 0 \\
& \quad -2v_1 + v_2 + w \geq 0 \\
& \quad -v_1 + 2v_3 + w \geq 0 \\
& \quad v_1 + v_2 + v_3 = 1 \\
& \quad v_1, \quad v_2, \quad v_3 \geq 0
\end{align*}
\]

- Homework: verify that these LPs are duals of each other