1. (40 points) Consider the following linear program.

\begin{align*}
\text{max} & \quad 2x_1 + 3x_2 - x_3 \\
\text{s.t.} & \quad x_1 + 2x_2 + x_3 = 5 \quad (a) \\
& \quad -x_1 + x_2 + x_3 \geq 1 \quad (b) \\
& \quad x_1 + x_2 + 2x_3 \leq 8 \quad (c) \\
& \quad x_1 \geq 0 \quad (d) \\
& \quad x_2 \geq 0 \quad (e)
\end{align*}

(a) (5 points) Which constraints are active at the point (2, 0, 3)? Use the letters next to each constraint to indicate your answer.

(b) (5 points) Is the vector $d^\top = (-1, 1, -1)$ a feasible direction at (2, 0, 3)? Justify your answer.

(c) (5 points) Is the vector $d^\top = (-1, 1, -1)$ an improving direction at (2, 0, 3)?

(d) (5 points) Could the point (2, 0, 3) be an optimal solution to the linear program? Justify your answer.
(e) (4 points) Is the point (2, 0, 3) a basic solution? Justify your answer.

(f) (4 points) Is the point (2, 0, 3) an extreme point? Justify your answer.

(g) (4 points) Is the point (2, 0, 3) degenerate?

(h) (6 points) Convert the linear program to canonical form.

(i) (2 points) Suppose the linear program was made nonlinear by changing the objective to

\[ \max x_1^2 + 3x_2^2 - x_3 \]

Is \( d^T = (-1, 1, -1) \) an improving direction at (2, 0, 3)? Justify your answer.
2. (30 points) Consider the following linear program:

\[
\text{max } 2x_1 + x_2 + 3x_3 - 20x_4 \\
\text{s.t. } x_1 - x_2 + 3x_3 + 5x_4 = 2 \\
\quad -x_1 + 2x_2 + 3x_3 + 4x_4 = 3 \\
x_1, x_2, x_3, x_4 \geq 0
\]

(a) (7 points) At the basic solution \((7, 5, 0, 0)\) what are the basic and nonbasic variables?

(b) (7 points) At the basic solution \((7, 5, 0, 0)\), one of the simplex directions is \((-9, -6, 1, 0)\). What is the other simplex direction? Are any of the directions improving?
(c) (7 points) At the basic solution \((0, \frac{1}{3}, \frac{7}{9}, 0)\), calculate the step size if the direction \(d^\top = (1, \frac{2}{3}, -\frac{1}{9}, 0)\) is used. Also, what are the entering and basic variables?

(d) (7 points) Write the Phase I linear program.

(e) (2 points) Professor May B. Wright says, “At the basic solution, \((0, 0, \frac{7}{3}, -1)\), we could compute Simplex directions even though it’s an infeasible direction! Then, we just choose an improving direction and see if we become feasible!” The simplex directions at this point are \((1, 0, 3, 2)\) and \((0, 1, -\frac{14}{3}, 3)\). What goes wrong if you try this?
3. (10 points) True or False. No justification needed.

______________To prove that the function \( f(x) = -|x| \) is not convex, we can simply note that
\[
f((1/3)1 + (2/3)(-1)) = -| -1/3 | = -1/3 > -1 = (1/3)f(1) + (2/3)f(-1).
\]

______________To prove that the set \( \{(x, y) : -1 \leq x - y \leq 1\} \) is convex we can simply note that
\[
f((1/3)1 + (2/3)(-1)) = -| -1/3 | = -1/3 > (1/3)f(1) + (2/3)f(-1) = -1.
\]

______________The set \( S = \{(x, y) : -1 \leq x - y \leq 1\} \), drawn in Figure 1, is the feasible region of a linear program but has no extreme points.

______________Converting the constraints that define \( S = \{(x, y) : -1 \leq x - y \leq 1\} \) (depicted in Figure 1) into canonical form results in the constraints
\[
\begin{align*}
x^+ - x^- - y^+ + y^- - s_1 &= -1 \\
x^+ - x^- - y^+ + y^- + s_2 &= 1 \\
x^+, x^-, y^+, y^-, s_1, s_2 &\geq 0.
\end{align*}
\]
A linear program using these constraints has no extreme points.

______________Let \( S = \{(x, y) : -1 \leq x - y \leq 1\} \) (depicted in Figure 1). The following optimization problem is unbounded:
\[
\max y - x \text{ s.t. } (x, y) \in S.
\]
Name (please print): ________________________

Only write your name above!

Instructions:

- No books are allowed. **One** 8.5 by 11 inch formula/note sheet is allowed.
- Show all work clearly. (little or no credit will be given for a numerical answer without the correct accompanying work. Partial credit is given where appropriate.)
- If you need more space than is provided, use the back of the previous page.
- Please read the question carefully. If you are not sure what a question is asking, ask for clarification.
- If you start over on a problem, please CLEARLY indicate what your final answer is, along with its accompanying work.

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