Lessons 24 + 25. The Simplex Method

1 Review

- Given an LP with $n$ decision variables, a solution $x$ is basic if:
  (a) it satisfies all equality constraints
  (b) at least $n$ linearly independent constraints are active at $x$

- A basic feasible solution (BFS) is a basic solution that satisfies all constraints of the LP.

- Canonical form LP:

  $$\begin{align*}
  \text{maximize} & \quad c^T x \\
  \text{subject to} & \quad Ax = b \\
  & \quad x \geq 0
  \end{align*}$$

  - $m$ equality constraints and $n$ decision variables (e.g. $A$ has $m$ rows and $n$ columns).
  - Standard assumptions: $m \leq n$, rank($A$) = $m$

- If $x$ is a basic solution of a canonical form LP, there exist $m$ basic variables of $x$ such that
  (a) the columns of $A$ corresponding to these $m$ variables are linearly independent
  (b) the other $n - m$ nonbasic variables are equal to 0

- The set of basic variables is the basis of $x$

2 Overview

- General improving search algorithm

  1. Find an initial feasible solution $x^0$
  2. Set $t = 0$
  3. while $x^t$ is not locally optimal do
  4. Determine a simultaneously improving and feasible direction $d$ at $x^t$
  5. Determine step size $\lambda$
  6. Compute new feasible solution $x^{t+1} = x^t + \lambda d$
  7. Set $t = t + 1$
  8. end while

- The simplex method is a specialized version of improving search

  - For canonical form LPs
  - Starts at a BFS in Step 1
  - Considers directions that point towards other BFSes in Step 4
  - Takes the maximum possible step size in Step 5
Example 1. Throughout this lesson, we will use the canonical form LP below:

\[
\begin{align*}
\text{maximize} & \quad 13x + 5y \\
\text{subject to} & \quad 4x + y + s_1 = 24 \\
& \quad x + 3y + s_2 = 24 \\
& \quad 3x + 2y + s_3 = 23 \\
& \quad x, y, s_1, s_2, s_3 \geq 0
\end{align*}
\]

3 Initial solutions

- For now, we will start by guessing an initial BFS

Example 2. Verify that \( x^0 = (0, 0, 24, 24, 23) \) is a BFS with basis \( B^0 = \{s_1, s_2, s_3\} \).

4 Finding feasible directions

- Two BFSes are adjacent if their bases differ by exactly 1 variable
- Suppose \( x^t \) is the current BFS with basis \( B^t \)
- Approach: consider directions that point towards BFSes adjacent to \( x^t \)
- To get a BFS adjacent to \( x^t \):
  - Put one nonbasic variable into \( B^t \)
  - Take one basic variable out of \( B^t \)
- Suppose we want to put nonbasic variable \( y \) into \( B^t \)
- This corresponds to the simplex direction \( d^y \) corresponding to nonbasic variable \( y \)
- \( d^y \) has a component for every decision variable
  - e.g. \( d^y = (d^y_x, d^y_y, d^y_{s_1}, d^y_{s_2}, d^y_{s_3}) \) for the LP in Example 1
- The components of the simplex direction \( d^y \) corresponding to nonbasic variable \( y \) are:
\( d_y^B = 1 \)
\( d_z^B = 0 \) for all other nonbasic variables \( z \)
\( d_w^B \) (uniquely) determined by \( Ad = 0 \) for all basic variables \( w \)

- Why does this work? Remember for LPs, \( d \) is a feasible direction at \( x \) if

\[
\begin{align*}
\mathbf{a}^T \mathbf{d} & \geq 0 & \text{for each active constraint of the form } & \mathbf{a}^T \mathbf{x} & \leq b
\end{align*}
\]

- Each nonbasic variable has a corresponding simplex direction

**Example 3.** The basis of the BFS \( x^0 = (0, 0, 24, 24, 23) \) is \( B^0 = \{s_1, s_2, s_3\} \). For each nonbasic variable, \( x \) and \( y \), we have a corresponding simplex direction. Compute the simplex directions \( d_x \) and \( d_y \).

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5 Finding improving directions

- Once we’ve computed the simplex direction for each nonbasic variable, which one do we choose?

- We choose a simplex direction \( d \) that is improving

- Recall that if \( f(x) \) is the objective function, \( d \) is an improving direction at \( x \) if

\[
\nabla f(x)^T \mathbf{d} \begin{cases} > 0 & \text{when maximizing } f \\ < 0 & \text{when minimizing } f \end{cases}
\]

- For LPs, \( f(x) = \mathbf{c}^T \mathbf{x} \), and so \( \nabla f(x) = \mathbf{c} \) for any \( x \)

- The **reduced cost** associated with nonbasic variable \( y \) is

\[
\bar{c}_y = \mathbf{c}^T d^y
\]

where \( d^y \) is the simplex direction associated with \( y \)

- The simplex direction \( d^y \) associated with nonbasic variable \( y \) is improving if

\[
\bar{c}_y \begin{cases} > 0 & \text{for a maximization LP} \\ < 0 & \text{for a minimization LP} \end{cases}
\]
Example 4. Consider the BFS $x^0 = (0, 0, 24, 24, 23)$ with basis $B^0 = \{s_1, s_2, s_3\}$. Compute the reduced costs $\bar{c}_x$ and $\bar{c}_y$ for nonbasic variables $x$ and $y$, respectively. Are $d^x$ and $d^y$ improving?

- If there is an improving simplex direction, we choose it
- If there is more than 1 improving simplex direction, we can choose any one of them
  - One option – **Dantzig’s rule**: choose the improving simplex direction with the most improving reduced cost (maximization LP – most positive, minimization LP – most negative)
- If there are no improving simplex directions, then the current BFS is a global optimal solution

6 Determining the maximum step size
- We’ve picked an improving simplex direction – how far can we go in that direction?
- Suppose $x^t$ is our current BFS, $d$ is the improving simplex direction we chose
- Our next solution is $x^{t+1} = x^t + \lambda d$ for some value of $\lambda \geq 0$
- How big can we make $\lambda$ while still remaining feasible?
- Recall that we computed $d$ so that $Ad = 0$
- $x^{t+1}$ satisfies the equality constraints $Ax = b$ no matter how large $\lambda$ gets, since
  $$Ax^{t+1} = A(x^t + \lambda d) = Ax^t + \lambda Ad = Ax^t = b$$
- So, the only thing that can go wrong are the nonnegativity constraints
  $$\Rightarrow$$ What is the largest $\lambda$ such that $x^{t+1} = x^t + \lambda d \geq 0$?

Example 5. Suppose we choose the improving simplex direction $d^x = (1, 0, -4, -1, 3)$. Compute the maximum step size $\lambda$ for which $x^1 = x^0 + \lambda d^x$ remains feasible.
• Note that only **negative** components of \( d \) determine maximum step size:

\[
\text{(nonnegative number)} + \lambda d \geq 0
\]

• The **minimum ratio test**: starting at the BFS \( x \), if any component of the improving simplex direction \( d \) is negative, then the maximum step size is

\[
\lambda_{\text{max}} = \min \left\{ \frac{x_j}{-d_j} : d_j < 0 \right\}
\]

**Example 6.** Verify that the minimum ratio test yields the same maximum step size you found in Example 5.

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<th>Example 6. Verify that the minimum ratio test yields the same maximum step size you found in Example 5.</th>
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• What if \( d \) has no negative components?

• For example:
  - Suppose \( x^0 = (0, 0, 1, 2, 3) \) is a BFS
  - \( d = (1, 0, 2, 4, 3) \) is an improving simplex direction at \( x \)
  - Then the next solution is
    \[
    x^1 = x^0 + \lambda d = (\lambda, 0, 1 + 2\lambda, 2 + 4\lambda, 3 + 3\lambda)
    \]
    - \( x^1 \geq 0 \) for all \( \lambda \geq 0 \)
    - We can improve our objective function and remain feasible forever!
    \[\Rightarrow\] The LP is unbounded

• **Test for unbounded LPs**: if all components of an improving simplex direction are nonnegative, then the LP is unbounded
7 Updating the basis

- We have our improving simplex direction \( d \) and step size \( \lambda_{\text{max}} \)
- We can compute our new solution \( x^{t+1} = x^t + \lambda_{\text{max}}d \)
- We also update the basis: update the set of basic variables

Example 7. Compute \( x^1 \). What is the basis \( B^1 \) of \( x^1 \)?

- Entering and leaving variables
  - The nonbasic variable corresponding to the chosen simplex direction enters the basis and becomes basic: this is the entering variable
  - Any one of the basic variables that define the maximum step size leaves the basis and becomes nonbasic: this is the leaving variable

8 Putting it all together: the simplex method

Step 0: Initialization. Identify a BFS \( x^0 \). Set solution index \( t = 0 \).

Step 1: Simplex directions. For each nonbasic variable \( y \), compute the corresponding simplex direction \( d^y \) and its reduced cost \( \bar{c}_y \).

Step 2: Check for optimality. If no simplex direction is improving, stop. The current solution \( x^t \) is optimal. Otherwise, choose any improving simplex direction \( d \). Let \( x_e \) denote the entering variable.

Step 3: Step size. If \( d \geq 0 \), stop. The LP is unbounded. Otherwise, choose the leaving variable \( x_\ell \) by computing the maximum step size \( \lambda_{\text{max}} \) according to the minimum ratio test.

Step 4: Update solution and basis. Compute the new solution \( x^{t+1} = x^t + \lambda_{\text{max}}d \). Replace \( x_\ell \) by \( x_e \) in the basis. Set \( t = t + 1 \). Go to Step 1.