8.11ab

(a) After converting the given LP into canonical form, we construct the Phase I LP as follows:

\[
\begin{align*}
\min & \quad a_1 + a_2 + a_3 \\
\text{s.t.} & \quad w_1 - 18w_2 + s_1 + a_1 + a_2 + a_3 = 9 \\
& \quad w_3 + w_4 - s_2 + a_2 = 14 \\
& \quad w_1 + w_2 - 2w_3 - 3w_4 + a_3 = 1 \\
& \quad w_1, \ w_2, \ w_3, \ w_4, \ s_1, \ s_2, \ a_1, \ a_2, \ a_3 \geq 0
\end{align*}
\]

(b) The decision variable vector is \( x = (w_1, w_2, w_3, w_4, s_1, s_2, a_1, a_2, a_3) \), the LHS matrix and the RHS vector are

\[
A = \begin{bmatrix}
1 & -18 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & -1 & 0 & 1 & 0 \\
1 & 1 & -2 & -3 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}, \quad b = \begin{bmatrix} 9 \\ 14 \end{bmatrix}
\]

and the objective function vector is \( c = (0, 0, 0, 0, 0, 1, 1, 1) \). The initial BFS and basis are

\[
x^0 = (0, 0, 0, 0, 0, 0, 9, 14, 1) \quad \text{T}, \quad B^0 = \{a_1, a_2, a_3\}.
\]

Iteration 1: The simplex directions are

\[
\begin{align*}
d^{w_1} &= (1, 0, 0, 0, 0, 0, -1, 0, -1) \quad \text{T}, \\
d^{w_2} &= (0, 1, 0, 0, 0, 0, 18, 0, -1) \quad \text{T}, \\
d^{w_3} &= (0, 0, 1, 0, 0, 0, 0, -1, 2) \quad \text{T}, \\
d^{w_4} &= (0, 0, 0, 1, 0, 0, 0, -1, 3) \quad \text{T}, \\
d^{s_1} &= (0, 0, 0, 0, 0, 1, -1, 0, 0) \quad \text{T}, \\
d^{s_2} &= (0, 0, 0, 0, 0, 1, 0, 1, 0) \quad \text{T},
\end{align*}
\]

and the reduced costs are

\[
\begin{align*}
\bar{c}_{w_1} &= c^T d^{w_1} = -2, & \bar{c}_{w_2} &= c^T d^{w_2} = 17, & \bar{c}_{w_3} &= c^T d^{w_3} = 1, & \bar{c}_{w_4} &= c^T d^{w_4} = 2, \\
\bar{c}_{s_1} &= c^T d^{s_1} = -1, & \bar{c}_{s_2} &= c^T d^{s_2} = 1.
\end{align*}
\]

So, using Dantzig’s rule, the entering variable is \( w_1 \). Using the minimum ratio test, the step size is

\[
\lambda_{\text{max}} = \min \left\{ \frac{9}{-(-1)}, \frac{1}{-(-1)} \right\} = 1,
\]

and so the leaving variable is \( a_3 \). Therefore, the next BFS and basis are

\[
x^1 = (1, 0, 0, 0, 0, 0, 8, 14, 0) \quad \text{T}, \quad B^1 = \{w_1, a_1, a_2\}.
\]

Iteration 2: The simplex directions are

\[
\begin{align*}
d^{w_2} &= (-1, 1, 0, 0, 0, 0, 19, 0, 0) \quad \text{T}, \\
d^{w_3} &= (2, 0, 1, 0, 0, 0, -2, -1, 0) \quad \text{T}, \\
d^{w_4} &= (3, 0, 0, 1, 0, 0, -3, -1, 0) \quad \text{T}, \\
d^{s_1} &= (0, 0, 0, 0, 1, 0, -1, 0, 0) \quad \text{T}, \\
d^{s_2} &= (0, 0, 0, 0, 0, 1, 0, 1, 0) \quad \text{T}, \\
d^{a_3} &= (-1, 0, 0, 0, 0, 0, 1, 0, 1) \quad \text{T},
\end{align*}
\]
and the reduced costs are
\[ \bar{c}_{w_2} = c^T d^{w_2} = 19, \quad \bar{c}_{w_3} = c^T d^{w_3} = -3, \quad \bar{c}_{w_4} = c^T d^{w_4} = -4, \quad \bar{c}_{x_1} = c^T d^{x_1} = -1, \]
\[ \bar{c}_{a_2} = c^T d^{a_2} = 1, \quad \bar{c}_{a_3} = c^T d^{a_3} = 2. \]

So, using Dantzig’s rule, the entering variable is \( w_4 \). Using the minimum ratio test, the step size is
\[ \lambda_{\text{max}} = \min \left\{ \frac{8}{-(-3)}, \frac{14}{-(-1)} \right\} = 8/3, \]
and so the leaving variable is \( a_1 \). Therefore, the next BFS and basis are
\[ x^2 = (9, 0, 0, 8/3, 0, 0, 0, 34/3, 0)^T, \quad B^2 = \{w_1, w_4, a_2\}. \]

**Iteration 3:** The simplex directions are
\[ d^{w_2} = (18, 1, 0, 19/3, 0, 0, 0, -19/3, 0)^T, \]
\[ d^{w_3} = (0, 0, 1, -2/3, 0, 0, 0, -1/3, 0)^T, \]
\[ d^{x_1} = (-1, 0, 0, -1/3, 1, 0, 0, 1/3, 0)^T, \]
\[ d^{a_2} = (0, 0, 0, 0, 1, 0, 0, -1, 3/0)^T, \]
\[ d^{a_3} = (0, 0, 0, 0, 0, 0, 0, -1, 3/0)^T, \]
and the reduced costs are
\[ \bar{c}_{w_2} = c^T d^{w_2} = -19/3, \quad \bar{c}_{w_3} = c^T d^{w_3} = -1/3, \quad \bar{c}_{x_1} = c^T d^{x_1} = 1/3, \quad \bar{c}_{a_2} = c^T d^{a_2} = 1, \]
\[ \bar{c}_{a_3} = c^T d^{a_3} = 2/3. \]

So, using Dantzig’s rule, the entering variable is \( w_2 \). Using the minimum ratio test, the step size is
\[ \lambda_{\text{max}} = \min \left\{ \frac{34/3}{-(-19/3)} \right\} = 34/19, \]
and so the leaving variable is \( a_2 \). Therefore, the next BFS and basis are
\[ x^3 = (783/19, 34/19, 0, 14, 0, 0, 0, 0, 0)^T, \quad B^3 = \{w_1, w_2, w_4\}. \]

**Iteration 4:** The simplex directions are
\[ d^{w_3} = (-18/19, -1/19, 1, -1, 0, 0, 0, 0, 0)^T, \]
\[ d^{x_1} = (-1/19, 1/19, 0, 0, 1, 0, 0, 0, 0)^T, \]
\[ d^{a_2} = (54/19, 34/19, 0, 1, 0, 1, 0, 0, 0)^T, \]
\[ d^{x_1} = (-1/19, 1/19, 0, 0, 0, 0, 1, 0, 0)^T, \]
\[ d^{a_2} = (-54/19, -34/19, 0, -1, 0, 0, 0, 1, 0)^T, \]
\[ d^{a_3} = (-18/19, -1/19, 0, 0, 0, 0, 0, 1, 0)^T, \]
and the reduced costs are
\[ \bar{c}_{w_3} = c^T d^{w_3} = 0, \quad \bar{c}_{x_1} = c^T d^{x_1} = 0, \quad \bar{c}_{a_2} = c^T d^{a_2} = 0, \quad \bar{c}_{a_3} = c^T d^{a_3} = 1, \]
\[ \bar{c}_{a_3} = c^T d^{a_3} = 1. \]

Since there are no improving simplex directions, the BFS \( x^3 \) is optimal.
Therefore, the optimal value of the Phase I LP is 0, and so \((783/19, 34/19, 0, 14, 0, 0)\) is a BFS to the original (canonical form) LP.