rock-paper-scissors

a. Let \( x_i \) = the probability that player R chooses action \( i \), for \( i \in \{ \text{Rock, Paper, Scissors} \} \). Then Player R's maximin strategy can be found with the following optimization model:

\[
\begin{align*}
\text{maximize} & \quad \min \{ x_2 - x_3, -x_1 + x_3, x_1 - x_2 \} \\
\text{subject to} & \quad x_1 + x_2 + x_3 = 1 \\
& \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0
\end{align*}
\]

This model is equivalent to the following linear program:

\[
\begin{align*}
\text{maximize} & \quad z \\
\text{subject to} & \quad z \leq x_2 - x_3 \\
& \quad z \leq -x_1 + x_3 \\
& \quad z \leq x_1 - x_2 \\
& \quad x_1 + x_2 + x_3 = 1 \\
& \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0
\end{align*}
\]

b. Let \( v_i \) = the probability the player C chooses action \( i \), for \( i \in \{ \text{Rock, Paper, Scissors} \} \). Then Player C's minimax strategy can be found with the following optimization model:

\[
\begin{align*}
\text{minimize} & \quad \max \{ -v_2 + v_3, v_1 - v_3, -v_1 + v_2 \} \\
\text{subject to} & \quad v_1 + v_2 + v_3 = 1 \\
& \quad v_1 \geq 0, v_2 \geq 0, v_3 \geq 0
\end{align*}
\]

This model is equivalent to the following linear program:

\[
\begin{align*}
\text{minimize} & \quad w \\
\text{subject to} & \quad w \geq -v_2 + v_3 \\
& \quad w \geq v_1 - v_3 \\
& \quad w \geq -v_1 + v_2 \\
& \quad v_1 + v_2 + v_3 = 1 \\
& \quad v_1 \geq 0, v_2 \geq 0, v_3 \geq 0
\end{align*}
\]

c. Let’s rewrite player R’s linear program so that all the decision variables are on the left hand side of the constraints, and all the constants are on the right hand side:

\[
\begin{align*}
\text{maximize} & \quad z \\
\text{subject to} & \quad z - x_2 + x_3 \leq 0 \quad (v_1) \\
& \quad z + x_1 - x_3 \leq 0 \quad (v_2) \\
& \quad z - x_1 + x_2 \leq 0 \quad (v_3) \\
& \quad x_1 + x_2 + x_3 = 1 \quad (w) \\
& \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0
\end{align*}
\]

The dual of this linear program is

\[
\begin{align*}
\text{minimize} & \quad w \\
\text{subject to} & \quad v_1 + v_2 + v_3 = 1 \\
& \quad v_2 - v_3 + w \geq 0 \\
& \quad -v_1 + v_3 + w \geq 0 \\
& \quad v_1 - v_2 + w \geq 0 \\
& \quad v_1 \geq 0, v_2 \geq 0, v_3 \geq 0, w \text{ free}
\end{align*}
\]
which is just player C’s linear program, rewritten so that all the decision variables are on the left hand side of the constraints, and all the constants are on the right hand side.

\[ \begin{align*}
\text{d. We can represent the “play each alternative with probability } \frac{1}{3} \text{” strategy for player R with the solution } & \quad x_1 = x_2 = x_3 = \frac{1}{3}. \text{ Note that this solution is feasible for player R’s linear program. The objective function value of this feasible solution is } \\
\min \{ & \frac{1}{3} - \frac{1}{3}, -\frac{1}{3} + \frac{1}{3}, \frac{1}{3} - \frac{1}{3} \} = 0.
\end{align*} \]

\[ \begin{align*}
\text{e. Similarly, we can represent the “play each alternative with probability } \frac{1}{3} \text{” strategy for player C with the solution } & \quad v_1 = v_2 = v_3 = \frac{1}{3}, \text{ which is feasible for player C’s linear program. The objective function value of this feasible solution is } \\
\max \{ & -\frac{1}{3} + \frac{1}{3}, \frac{1}{3} - \frac{1}{3}, -\frac{1}{3} + \frac{1}{3} \} = 0.
\end{align*} \]

f. Note that we have feasible solutions to player R’s and player C’s linear programs with equal objective function values. Since these linear programs form a primal-dual pair, by weak duality, these solutions must be optimal. Therefore, we can conclude that the “play each alternative with probability } \frac{1}{3} \text{” strategy is optimal for both player R and player C.