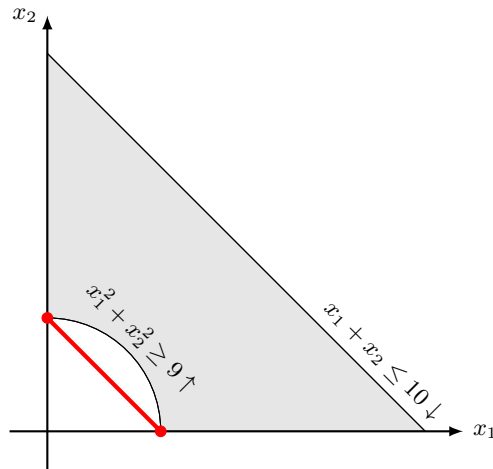


6.14

(a) The region is not convex. Since the region is in \mathbb{R}^2 , we can see this visually by graphing the region:



Observe that $(3, 0)$ and $(0, 3)$ are both points in the feasible region, but there exist points on the line segment connecting these two points are not in the feasible region. Therefore, the region is not convex. To show this formally, let $\mathbf{x}^\top = (3, 0)$ and $\mathbf{y}^\top = (0, 3)$. Algebraically, \mathbf{x} and \mathbf{y} are both points in the region as they are both nonnegative and

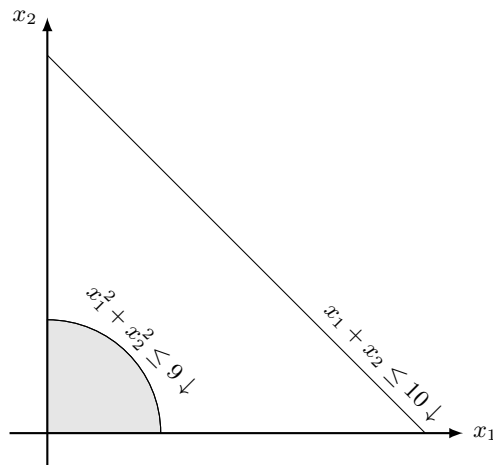
$$\begin{aligned}x_1^2 + x_2^2 &= (3)^2 + (0)^2 = 9 \\x_1 + x_2 &= (3) + (0) = 3 \leq 10 \\y_1^2 + y_2^2 &= (0)^2 + (3)^2 = 9 \\y_1 + y_2 &= (0) + (3) = 3 \leq 10\end{aligned}$$

However, using $\lambda = 1/2$, we have that

$$\lambda \mathbf{x} + (1 - \lambda) \mathbf{y} = 1/2 \begin{pmatrix} 3 \\ 0 \end{pmatrix} + 1/2 \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix}$$

and because $(3/2)^2 + (3/2)^2 = 18/4 = 9/2 < 9$, $\lambda \mathbf{x} + (1 - \lambda) \mathbf{y}$ is not in the region. Therefore, the region is not convex.

(b) The region is convex. Again, since the region is in \mathbb{R}^2 , we can see this visually by graphing the region:



Observe that for any two points in the region, the line segment connecting these two points lies completely in the region. Therefore, the region is convex.

To prove this formally, we must show that for arbitrary points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ that satisfy the given constraints and $\lambda \in [0, 1]$, the point $\mathbf{z} = \lambda\mathbf{x} + (1 - \lambda)\mathbf{y}$ satisfies all the constraints. Because both \mathbf{x} and \mathbf{y} are nonnegative and $0 \leq \lambda \leq 1$, \mathbf{z} is also nonnegative. Also, note that

$$\begin{aligned} z_1 + z_2 &= (\lambda x_1 + (1 - \lambda)y_1) + (\lambda x_2 + (1 - \lambda)y_2) \\ &= \lambda(x_1 + x_2) + (1 - \lambda)(y_1 + y_2) \\ &\leq \lambda(10) + (1 - \lambda)(10) = 10 \end{aligned}$$

where the last inequality follows because both \mathbf{x} and \mathbf{y} satisfy the constraints (in particular, the second constraint). Based on Example 6.21, we have that the function $f(\mathbf{x}) = x_1^2 + x_2^2$ is convex which means that

$$\begin{aligned} f(\mathbf{z}) &= f(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) \\ &\leq \lambda f(\mathbf{x}) + (1 - \lambda)f(\mathbf{y}) \\ &= \lambda(x_1^2 + x_2^2) + (1 - \lambda)(y_1^2 + y_2^2) \\ &\leq \lambda 9 + (1 - \lambda)9 = 9 \end{aligned}$$

so \mathbf{z} satisfies the first constraint. Thus, \mathbf{z} satisfies all constraints. Therefore, the region is convex.

- (c) The region is convex: it is the feasible region of a linear program. To show this formally, let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ satisfy all the constraints and $\lambda \in [0, 1]$ all be arbitrarily chosen. Then, as in (b), let $\mathbf{z} = \lambda\mathbf{x} + (1 - \lambda)\mathbf{y}$. Then

$$\begin{aligned} z_1 + z_2 + 3z_3 &= (\lambda x_1 + (1 - \lambda)y_1) + (\lambda x_2 + (1 - \lambda)y_2) + 3(\lambda x_3 + (1 - \lambda)y_3) \\ &= \lambda(x_1 + x_2 + 3x_3) + (1 - \lambda)(y_1 + y_2 + 3y_3) \\ &\leq \lambda 10 + (1 - \lambda)10 = 10 \end{aligned}$$

Similarly,

$$\begin{aligned} 2z_1 + 3z_2 + z_3 &= 2(\lambda x_1 + (1 - \lambda)y_1) + 3(\lambda x_2 + (1 - \lambda)y_2) + (\lambda x_3 + (1 - \lambda)y_3) \\ &= \lambda(2x_1 + 3x_2 + x_3) + (1 - \lambda)(2y_1 + 3y_2 + y_3) \\ &\geq \lambda 5 + (1 - \lambda)5 = 5 \end{aligned}$$

Finally, $\mathbf{z} = \lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \geq \lambda \cdot 0 + (1 - \lambda) \cdot 0 = 0$. Thus, \mathbf{z} satisfies all constraints. Therefore, the region is convex.

- (d) The region is not convex. Consider the points

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 5 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{y} = \begin{pmatrix} 10 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

which both satisfy all the constraints. However, when $\lambda = 1/2$,

$$\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} = \begin{pmatrix} 5 \\ 0 \\ 5/2 \\ 0 \end{pmatrix}$$

is not a vector of integers and so, is not in the region. Therefore the region is not convex.