8.1

After adding slack variables, the decision variable vector and the LHS constraint matrix are:

$$\mathbf{x} = (x_1, x_2, x_3, s_1, s_2, s_3)^\top \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 6 & 6 & 2 & 0 & 0 & 1 \end{bmatrix}.$$

The objective function vector is $\mathbf{c} = (8, 9, 5, 0, 0, 0)^\top$. The initial BFS and basis are

$$\mathbf{x}^0 = (0, 0, 0, 2, 3, 8)^\top, \qquad \mathcal{B}^0 = \{s_1, s_2, s_3\}.$$

Iteration 1: First, we compute simplex directions for every nonbasic variable. For x_1, we solve $A\mathbf{d}^{x_1} = \mathbf{0}$, i.e.,

$$\begin{aligned} 1 + d_{s_1}^{x_1} &= 0 \\ 2 + d_{s_2}^{x_1} &= 0 \\ 6 + d_{s_3}^{x_1} &= 0 \end{aligned}$$

so $\mathbf{d}^{x_1} = (1, 0, 0, -1, -2, -6)^\top$. Similar calculations show that

$$\begin{aligned} \mathbf{d}^{x_2} &= (0, 1, 0, -1, -3, -6)^\top, \\ \mathbf{d}^{x_3} &= (0, 0, 1, -2, -4, -2)^\top. \end{aligned}$$

The reduced costs are

$$\bar{c}_{x_1} = \mathbf{c}^\top \mathbf{d}^{x_1} = 8, \qquad \bar{c}_{x_2} = \mathbf{c}^\top \mathbf{d}^{x_2} = 9, \qquad \bar{c}_{x_3} = \mathbf{c}^\top \mathbf{d}^{x_3} = 5$$

so we choose x_2 as our entering variable. To calculate step size, we use the minimum ratio test:

$$\lambda_{\max} = \min\left\{ \frac{x_j}{-d_j} : d_j < 0 \right\} = \min\left\{ \frac{2}{-(-1)}, \frac{3}{-(-3)}, \frac{8}{-(-6)} \right\} = \min\{2, 1, 4/3\} = 1.$$

Thus s_2 is our leaving variable, and the new solution and basis are

$$\mathbf{x}^1 = (0, 1, 0, 1, 0, 2)^\top, \qquad \mathcal{B}^1 = \{x_2, s_1, s_3\}.$$

Iteration 2: Again, we compute simplex directions for each of the nonbasic variables. We see that the directions are

$$\begin{aligned} \mathbf{d}^{x_1} &= (1, -2/3, 0, -1/3, 0, -2)^\top, \\ \mathbf{d}^{x_3} &= (0, -4/3, 1, -2/3, 0, 6)^\top, \\ \mathbf{d}^{s_2} &= (0, -1/3, 0, 1/3, 1, 2)^\top. \end{aligned}$$

The reduced costs are

$$\bar{c}_{x_1} = \mathbf{c}^\top \mathbf{d}^{x_1} = 2, \qquad \bar{c}_{x_3} = \mathbf{c}^\top \mathbf{d}^{x_3} = -7, \qquad \bar{c}_{s_2} = \mathbf{c}^\top \mathbf{d}^{s_2} = -3.$$

The entering variable is x_1. The step size is $\lambda_{\max} = 1$. The leaving variable is s_3. The new solution and basis are

$$\mathbf{x}^2 = (1, 1/3, 0, 2/3, 0, 0)^\top, \qquad \mathcal{B}^2 = \{x_1, x_2, s_1\}.$$

Iteration 3: Again, we compute simplex directions for each nonbasic variable. The directions are

$$\begin{aligned} \mathbf{d}^{x_3} &= (3, -10/3, 1, -5/3, 0, 0)^\top, \\ \mathbf{d}^{s_2} &= (1, -1, 0, 0, 1, 0)^\top, \\ \mathbf{d}^{s_3} &= (-1/2, 1/3, 0, 1/6, 0, 1). \end{aligned}$$

The reduced costs are

$$\bar{c}_{x_3} = \mathbf{c}^\top \mathbf{d}^{x_3} = -1, \qquad \bar{c}_{s_2} = \mathbf{c}^\top \mathbf{d}^{s_2} = -1, \qquad \bar{c}_{s_3} = \mathbf{c}^\top \mathbf{d}^{s_3} = -1$$

Because none of the simplex directions are improving, the solution $\mathbf{x}^2$ is optimal.