We now describe the local $p$-Median problem, a facility location problem that is similar to the general $p$-Median problem, formulated in the text on p. 136 as Integer Program 4.5, but has a crucial difference in that demand can only be satisfied "locally". Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a given undirected graph. The crucial assumption is that a demand node $i \in \mathcal{V}$ can only be served by a facility at node $j \in \mathcal{V}$ if either $(j, i) = (i, j) \in \mathcal{E}$, or $i = j$.

Also given are:

- a set of demand nodes, denoted by $I \subseteq \mathcal{V}$;
- a population at each demand node, denoted by $h_i$;
- a set of possible location nodes where facilities might be built, denoted by $J \subseteq \mathcal{V}$;
- a positive integer $p$, which is the maximum number of facilities that can be built;
- a minimum amount of population whose demand must be satisfied, denoted by $T$;
- arc distances, denoted by $c_{ij}$; and
- for each $i \in I$, a set of neighborhood nodes, denoted by $N_i$, and defined formally by
  \[ N_i = \{ j \in J | (i, j) = (j, i) \in \mathcal{E} \text{ or } i = j \}, \]
  i.e., $N_i$ consists of location nodes that can serve $i$.

We note that $c_{ij}$ are different then the book’s $d_{ij}$ as $d_{ij}$ represent shortest path distances, whereas $c_{ij}$ are arc distances (and may even be greater than the shortest path distances).

The problem is as follows: Where should the $p$ facilities be located so as to minimize the total population-weighted distance between demand nodes served and facilities? For the formulation, we use two sets of binary variables,

\[
y_{ij} = \begin{cases} 
1 & \text{if } i \text{ is served by } j \\
0 & \text{otherwise.}
\end{cases}
\]

for every demand node $i$ and location node $j \in N_i$, and

\[
x_j = \begin{cases} 
1 & \text{if a facility is open at location node } j \\
0 & \text{otherwise.}
\end{cases}
\]

for every location node $j \in J$. The formulation is

\[
\begin{align*}
\text{min} & \quad \sum_{i \in I} h_i \left( \sum_{j \in N_i} c_{ij} y_{ij} \right) \\
\text{s.t.} & \quad \sum_{j \in J} x_j \leq p \\
& \quad y_{ij} \leq x_j \quad \forall i \in I, \forall j \in N_i \\
& \quad \sum_{i \in I} h_i \left( \sum_{j \in N_i} y_{ij} \right) \geq T \\
& \quad \sum_{j \in N_i} y_{ij} \leq 1 \quad \forall i \in I \\
& \quad x_j, y_{ij} \in \{0, 1\} \quad \forall i \in I, \forall j \in N_i.
\end{align*}
\]

The objective, (a), multiplies the population by the distance from the facility location used (if any) and adds these products to calculate the population-weighted sum of distances. Note that the order of summation matters as $N_i$ depends on the index $i$, which is also true in the constraints (c), (d), and (f). The upper bound on facilities is enforced in constraint (b) – note that in the book, the assumption is that exactly $p$ facilities must be opened. Constraints (c) enforce that if demand node $i$ is satisfied by a facility at location node $j$, then a facility must be open at location $j$. The lower bound on demand satisfied is enforced by constraint (d). Constraints (e) ensure that at most one facility is counted as serving a given demand node. Constraints (f) enforce that the variables are binary.
Some notes on the formulation and problem:

- The crucial differences between the local $p$-Median problem and the $p$-Median problem are that in the local problem a facility at location node $j$ can only serve demand node $i$ if $(i, j) = (j, i) \in E$ or $i = j$. In the general $p$-Median problem, all demand nodes can be served from any location node so long as a facility is open there. Thus, in the general $p$-Median, the book assumes every demand node is satisfied. In the local $p$-Median, we assume that a minimum amount of demand must be satisfied.

- Note that bounds on meaningful $p$ and $T$ can be determined from the Set-Cover Location problem (p. 132, Integer Program 4.2), and Maximal Covering Location problem (p. 133, Integer Program 4.3), respectively.