SA421 – Simulation Modeling Fall 2013
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SA421: Variance identities

Let X and Y be random variables and c is some real constant. In this note, we are interested in calculating $\text{Var}(X+Y)$ and $\text{Var}(cX)$. Recall that

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \text{ and } \text{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

Then,

$$\begin{aligned}\text{Var}(X+Y) = \mathbb{E}(X+Y)^2 - (\mathbb{E}[X+Y])^2 &= \mathbb{E}[X^2 + 2XY + Y^2] - (\mathbb{E}[X])^2 - 2\mathbb{E}[X]\mathbb{E}[Y] - (\mathbb{E}[Y])^2 \\ &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 + \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 + 2\mathbb{E}[XY] - 2\mathbb{E}[X]\mathbb{E}[Y] \\ &= \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X,Y).\end{aligned}$$

Now consider

$$\text{Var}(cX) = \mathbb{E}[(cX)^2] - (\mathbb{E}[cX])^2 = \mathbb{E}[c^2X^2] - (c\mathbb{E}[X])^2 = c^2(\mathbb{E}[X^2] - (\mathbb{E}[X])^2) = c^2\text{Var}(X).$$

To summarize, we can state that:

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X,Y) \text{ and } \text{Var}(cX) = c^2\text{Var}(X). \tag{1}$$