Math 431 – Spring 2014

Homework 9

Due: April 17th (Sections 2 and 4) or 18th (Sections 1 and 3), 2014, depending upon your section (according to the instructions of your lecturer)

PLEASE READ THE INSTRUCTIONS/SUGGESTIONS ON THE COURSE WEBPAGE.

Hand in the following problems:

1. Let $E[X] = 1$, $E[X^2] = 3$, $E[XY] = -4$ and $E[Y] = 2$. Find $\text{Cov}(X, 2X + Y)$.

   Solution: By the definition of covariance and linearity of expectation,

$$\begin{aligned}\text{Cov}(X, 2X+Y) =& E[X(2X+Y)] - E[X]E[2X+Y] \\ =& 2E[X^2] + E[XY] - E[X](2E[X] + E[Y]) \\ =& 2(3) + (-4) - 1(2(1) + 2) \\ =& -2.\end{aligned}$$

2. Suppose you roll a fair 20-sided die 5 times. Let X denote the different outcomes you see. (For example (20, 17,18,17,3) would be $X = 4$).

   (a) Find the mean and variance of X.

   Solution: We compute the mean and variance much like the die problem from homework 8. Let X_i denote the indicator function of the first time i is seen on one of the five die rolls. Then, by linearity of expectation and exchangeability,

$$\begin{aligned}E[X] =& E[X_1] + \cdots + E[X_20] \\ =& 20E[X_1] \\ =& 20P(\text{number 1 is seen at least once in the five rolls}) \\ =& 20(1 - P(\text{number 1 is not seen in the five rolls})) \\ =& 20\left(1 - \left(\frac{19}{20}\right)^5\right).\end{aligned}$$

   To find the variance, we use fact 7.22,

$$\begin{aligned}Var(X) =& Var\left(\sum_{i=1}^{20} X_i\right) \\ =& \sum_{i=1}^{20} Var(X_i) + 2\sum_{i<j} Cov(X_i, X_j).\end{aligned}$$

   By exchangeability,

$$Var(X) = 20Var(X_1) + 20 \cdot 19Cov(X_1, X_2).$$

For the indicator function,

$$Var(X_1) = p(1-p) = \left(1 - \left(\frac{19}{20}\right)^5\right)\left(\frac{19}{20}\right)^5.$$

The covariance is, by exchangeability and $X_1X_2 = X_{1\cap 2}$,

$$\begin{aligned}
Cov(X_1, X_2) =& E[X_1X_2] - E[X_1]E[X_2] = E[X_{1\cap 2}] - E[X_1]^2 \\
=& P(1 \text{ and } 2 \text{ are seen at least once in the 5 rolls}) \\
-& P(1 \text{ is seen at least once in the 5 rolls})^2 \\
=& (1 - P(1 \text{ or } 2 \text{ are not seen in the 5 rolls})) \\
-& (1 - P(1 \text{ is not seen in the 5 rolls}))^2 \\
=& \left(1 - \left(\frac{18}{20}\right)^5\right) - \left(1 - \left(\frac{19}{20}\right)^5\right)^2.
\end{aligned}$$

Putting it all together we have,

$$Var(X) = 20\left(1 - \left(\frac{19}{20}\right)^5\right)\left(\frac{19}{20}\right)^5 + 20 \cdot 19\left[\left(1 - \left(\frac{18}{20}\right)^5\right) + \left(1 - \left(\frac{19}{20}\right)^5\right)^2\right].$$

(b) What is the mean and variance if you roll the die n times?

Solution: In both the mean and variance, the only numbers that depend on the number of rolls are the fives in the exponent. Thus,

$$E[X] = 20E[X_1] = 20\left(1 - \left(\frac{19}{20}\right)^n\right) \to 20$$

as $n \to \infty$. We would expect to see all 20 numbers show up if we are rolling many many times. We would expect the variance to tend to zero. We get,

$$Var(X) = 20\left(1 - \left(\frac{19}{20}\right)^n\right)\left(\frac{19}{20}\right)^n + 20 \cdot 19\left[\left(1 - \left(\frac{18}{20}\right)^n\right) - \left(1 - \left(\frac{19}{20}\right)^n\right)^2\right]$$

As $n \to \infty$ we get,

$$Var(X) = 20(1-0)(0) + 20 \cdot 19(1-1) = 0$$

3. Let $Z_1, Z_2, \ldots, Z_n$ be independent normal random variables with mean 0 and variance 1. Let

$$\chi = Z_1^2 + \cdots + Z_n^2.$$

(a) Using that χ is the sum of independent random variables, compute both the mean and variance of χ.

Solution: For the mean we use linearity of expectation and the fact that the Z_i are identically distributed. Furthermore $E[Z_i^2] = E[Z_i^2] - E[Z_i]^2 = Var(Z) = 1$, because $E[Z_i] = 0$. Therefore we have,

$$E[\chi] = \sum_{i=1}^{n} E[Z_i^2] = nE[Z_1^2] = n.$$

For the variance, by independence,

$$Var(\chi) = \sum_{i=1}^{n} Var(Z_i^2) = nVar(Z_1^2).$$

The variance is defined as

$$Var(Z_1^2) = E[Z_1^4] - E[Z_1^2]^2.$$

We computed the fourth moment of a standard normal random variable in homework 4 (number 4), $E[Z_1^4] = 3$. Thus,

$$Var(\chi) = nVar(Z_1^2) = n(3-1) = 2n.$$

(b) Find the moment generating function of χ and use it to compute the mean and variance of χ.
Solution: The moment generating function of χ is defined to be

$$E[e^{t\chi}] = E[e^{t(Z_1^2+Z_2^2+\cdots+Z_n^2)}].$$

By independence of Z_i we use fact 7.13, to write the right hand side as a product of moment generating function. Since the Z_i are identically distributed, then it is a product of the same moment generating function. That is,

$$M_\chi(t) = E[e^{t\chi}] = E[e^{t(Z_1^2)}]^n = M_{Z_1^2}(t)^n.$$

We now compute the moment generating function of Z_1^2 using the density of the standard normal distribution. We have

$$\begin{aligned} E[e^{tZ_1^2}] =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz^2} e^{-z^2/2}\, dz \\ =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(2t-1)z^2/2}\, dz. \end{aligned}$$

This integral convergences only for $t < 1/2$. Using the substitution $\tilde{z} = \sqrt{1-2t}z$ we have

$$\begin{aligned} E[e^{tZ_1^2}] =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(2t-1)z^2/2}\, dz \\ =& \frac{1}{\sqrt{1-2t}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\tilde{z}^2/2}\, d\tilde{z} \\ =& \frac{1}{\sqrt{1-2t}} \quad \text{for } t < 1/2. \end{aligned}$$

Therefore,

$$M_\chi(t) = \begin{cases} (1-2t)^{-n/2} & \text{for } t < 1/2 \\ \infty & \text{for } t \geq 1/2. \end{cases}$$

Using the moment generating function we calculate the mean to be

$$E[\chi] = M_\chi'(0) = n.$$

For the variance, we calculate the second moment,

$$E[\chi^2] = M''_\chi(0) = n(n-2) = n(n-2).$$

The variance is

$$Var(\chi) = E[\chi^2] - E[\chi]^2 = n^2 - 2n - n^2 = 2n.$$

4. Let (X, Y) be a uniformly distributed random point on the quadrilateral D with vertices $(0,0), (2,0), (1,1)$, and $(0,1)$. Calculate the covariance of X and Y.

   Solution: To calculate the covariance we need to calculate

$$E[XY], \quad E[X], \quad E[Y].$$

   First the joint distribution of (X, Y) on the quadrilateral, noting that the area is $3/2$, is

$$p_{X,Y}(x,y) = \begin{cases} \frac{2}{3}, & (x,y) \in D \\ 0, & (x,y) \notin D. \end{cases}$$

   Then integrating we have,

$$\begin{aligned} E[XY] &= \int_0^1 \int_0^{2-y} \frac{2}{3} xy \, dx \, dy \\ &= \int_0^1 \frac{2}{6} y(2-y)^2 \\ &= \frac{2}{6} \left(\frac{4}{2} y^2 - \frac{4}{3} y^3 + \frac{1}{4} y^4 \right)_0^1 \\ &= \frac{2}{6} \cdot \frac{11}{12} = \frac{11}{36}. \end{aligned}$$

$$\begin{aligned} E[X] &= \int_0^1 \int_0^{2-y} \frac{2}{3} x \, dx \, dy \\ &= \int_0^1 \frac{2}{6} (2-y)^2 \\ &= \frac{2}{6} \left(4y - \frac{4}{2} y^2 + \frac{1}{3} y^3 \right)_0^1 \\ &= \frac{2}{6} \cdot \frac{7}{3} = \frac{7}{9}. \end{aligned}$$

$$\begin{aligned} E[Y] &= \int_0^1 \int_0^{2-y} \frac{2}{3} y \, dx \, dy \\ &= \int_0^1 \frac{2}{6} (2-y) y \\ &= \frac{2}{3} \left(\frac{2}{2} y^2 - \frac{1}{3} y^3 \right)_0^1 \\ &= \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}. \end{aligned}$$

By the definition of covariance, we have

$$Cov(X,Y) = E[XY] - E[X]E[Y] = \frac{11}{36} - \frac{7}{9}\cdot\frac{4}{9} = \frac{11}{36} - \frac{28}{81}.$$

5. Let the joint pmf of (X, Y) be given by the table below.

X \ Y	0	1	2	3
1	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{1}{15}$
2	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{10}$
3	$\frac{1}{30}$	$\frac{1}{30}$	0	$\frac{1}{10}$

(a) Find Cov(X, Y).

Solution: To find the covariance, we compute

$$E[XY], \quad E[X], \quad E[Y].$$

Thus we have

$$\begin{aligned}
E[XY] =& \sum_{i=1}^{3}\sum_{j=0}^{3} ijp(x=i, y=j) \\
=& 0\left(\frac{1}{15}+\frac{1}{10}+\frac{1}{30}\right) + 1\cdot\frac{1}{15} + 2\cdot\left(\frac{1}{10}+\frac{2}{15}\right) \\
&+3\cdot\left(\frac{1}{30}+\frac{1}{15}\right) + 4\left(\frac{1}{5}\right) + 6\left(\frac{1}{10}\right) \\
\approx& 2.233
\end{aligned}$$

For $E[X]$ and $E[Y]$ we can use the marginal pmfs, by summing over the columns or rows, to simplify the computations.

X	$f_X(x)$
1	$\frac{1}{3}$
2	$\frac{1}{2}$
3	$\frac{1}{6}$

$f_Y(y)$ \ Y	0	1	2	3
1	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{4}{15}$

$$
\begin{aligned}
E[X] =& \sum_{i=1}^{3}\sum_{j=0}^{3} ip(x=i,y=j) \\
=& \sum_{i=1}^{3} ip_X(i) \\
=& \frac{1}{3} + 2\cdot\frac{1}{2} + 3\cdot\frac{1}{6} \\
=& \frac{11}{6} \\
\approx& 1.833
\end{aligned}
$$

$$
\begin{aligned}
E[Y] =& \sum_{j=0}^{3}\sum_{i=1}^{3} jp(x=i,y=j) \\
=& \sum_{j=0}^{3} jp_Y(j) \\
=& 0\cdot\frac{1}{5} + 1\cdot\frac{1}{5} + 2\cdot\frac{1}{3} + 3\cdot\frac{4}{15} \\
\approx& 1.667
\end{aligned}
$$

Using the above calculations, the covariance is,

$$Cov(X,Y) = E[XY] - E[X]E[Y] = 2.233 - (1.833)(1.66)$$

(b) Find $\text{Corr}(X,Y)$.

Solution: To find the correlation coefficient, we need the covariance from above as well as the variance of X and Y. To find the variance, we need the second moments.

$$
\begin{aligned}
E[X^2] =& \sum_{i=1}^{3}\sum_{j=0}^{3} i^2p(x=i,y=j) \\
=& \sum_{i=1}^{3} i^2p_X(i) \\
=& \frac{1}{3} + 2^2\cdot\frac{1}{2} + 3^2\cdot\frac{1}{6}
\end{aligned}
$$

$$
\begin{aligned}
E[Y^2] =& \sum_{j=0}^{3}\sum_{i=1}^{3} j^2p(x=i,y=j) \\
=& \sum_{j=0}^{3} j^2p_Y(j) \\
=& 0^2\cdot\frac{1}{5} + 1^2\cdot\frac{1}{5} + 2^2\cdot\frac{1}{3} + 3^2\cdot\frac{4}{.}
\end{aligned}
$$

The correlation coefficient is defined as,

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} = \frac{2.233 - (1.833)(1.66)}{\sqrt{\ldots\ldots\ldots}\sqrt{\ldots\ldots}}$$

6. Let X be uniformly distributed on $[-a, a]$ for $a > 0$ and $Y = X^2$. Show that X and Y are uncorrelated, even though Y is a function of X.

   Solution: X is uniformly distributed on $[-a, a]$ and therefore has the probability density function

$$p_X(x) = \begin{cases} \frac{1}{2a}, & x \in [-a, a] \\ 0, & x \notin [-a, a]. \end{cases}$$

   To show that X and Y are uncorrelated, we must show that $Cov(X, Y) = 0$, or

$$Cov(X,Y) = E[XY] - E[X]E[Y] = E[X^3] - E[X]E[X^2] = 0$$

   We compute the third moment of X using the density function,

$$\begin{aligned} E[X^3] &= \int_{-\infty}^{\infty} x^3 p_X(x)\, dx \\ &= \int_{-a}^{a} \frac{x^3}{2a}\, dx \\ &= \frac{(a)^4 - (-a)^4}{8a} \\ &= 0. \end{aligned}$$

   Because $1/2a$ is constant in x, and therefore symmetric about $x = 0$, then every odd moment of X will be zero. That is,

$$E[X] = E[X^3] = \cdots = E[X^{2n+1}] = 0$$

   for every $n = 0, 1, 2, 3, \ldots$. Thus

$$Cov(X,Y) = E[XY] - E[X]E[Y] = E[X^3] - E[X]E[X^2] = 0 - 0 \cdot E[X^2] = 0.$$

   Therefore, X and Y are uncorrelated.

7. Let I_A be the indicator of the event A. Show that for any A, B we have

$$\text{Corr}(I_A, I_B) = \text{Corr}(I_{A^c}, I_{B^c}).$$

   Solution: We start with the left hand side of the above equation. We start with the definition of covariance,

$$Cov(I_A, I_B) = E[I_A I_B] - E[I_A]E[I_B] = P(A \cap B) - P(A)P(B).$$

   Next we use the identity $1 - P(A^c \cap B^c) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$. This yields,

$$Cov(I_A, I_B) = P(A) + P(B) - (1 - P(A^c \cap B^c)) - P(A)P(B).$$

Next we substitute $P(A) = 1 - P(A^c)$ and $P(B) = 1 - P(B^c)$ to get

$$Cov(I_A, I_B) = P(A) + P(B) - P(A^c \cap B^c) - (1 - P(A^c))(1 - P(B^c)).$$

Simplifying yields,

$$Cov(I_A, I_B) = P(A^c \cap B^c) - P(A^c)P(B^c) = E[I_{A^c} I_{B^c}] - E[I_{A^c}]E[I_{B^c}] = Cov(I_{A^c}, I_{B^c})$$

To finish the problem, we must show that the variances are equal. For any indicator function we have

$$Var(I_A) = E[I_A^2] - E[I_A]^2 = P(A) - P(A)^2 = P(A)(1 - P(A)) = P(A)P(A^c).$$

Thus

$$Var(I_A) = Var(I_{A^c}),$$

and therefore,

$$Corr(I_A, I_B) = Corr(I_{A^c}, I_{B^c}).$$

8. Suppose that for the random variable X, Y we have $E[X] = 2$, $E[Y] = 1$, $E[X^2] = 5$, $E[Y^2] = 10$ and $E[XY] = 1$.

   (a) Compute $\text{Corr}(X, Y)$.
   Solution: First we compute the covariance,

   $$Cov(X, Y) = E[XY] - E[X]E[Y] = 1 - (2)(1) = -1.$$

   Now the variances,

   $$Var(X) = E[X^2] - E[X]^2 = 5 - 4 = 1$$
   $$Var(Y) = E[Y^2] - E[Y]^2 = 10 - 1 = 9.$$

   Now the correlation is,

   $$Corr(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} = \frac{-1}{\sqrt{1}\sqrt{9}} = -\frac{1}{3}.$$

   (b) Find a number c so that X and $X + cY$ are uncorrelated.
   Solution: We want to find a c such that the covariance is zero. That is,

   $$\begin{aligned} Cov(X, X + cY) =& E[X(X + cY)] - E[X]E[X + cY] \\ =& E[X^2] + cE[XY] - E[X](E[X] + cE[Y]) \\ =& 5 + c - 2(2 + c) \\ =& 1 - c \end{aligned}$$

   Thus we solve the equation $1 - c = 0$ to find $c = 1$ will result in X and $X - Y$ being uncorrelated.

9. Let the joint probability density function of (X, Y) be given by

$$f(x, y) = \begin{cases} \left(\frac{1}{10}\right)\left(\frac{1}{x}\right) & \text{for } 0 \le x \le 10,\ 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}$$

   Find:

(a) Cov(X,Y).

Solution: To find the covariance we compute

$$E[XY], \quad E[X], \quad E[Y].$$

We must watch our limits of integration.

$$\begin{aligned} E[XY] &= \int_0^{10}\int_0^x \frac{1}{10x} xy\, dy\, dx \\ &= \int_0^{10} \frac{x^2}{20}\, dx \\ &= \frac{10^3}{60}. \end{aligned}$$

$$\begin{aligned} E[X] &= \int_0^{10}\int_0^x \frac{1}{10x} x\, dy\, dx \\ &= \int_0^{10} \frac{x}{10}\, dx \\ &= \frac{10^2}{20}. \end{aligned}$$

$$\begin{aligned} E[Y] &= \int_0^{10}\int_0^x \frac{1}{10x} y\, dy\, dx \\ &= \int_0^{10} \frac{x}{20}\, dx \\ &= \frac{10^2}{40}. \end{aligned}$$

Thus the covariance is,

$$Cov(X,Y) = E[XY] - E[X]E[Y] = \frac{10^3}{60} - \frac{10^2}{20}\frac{10^2}{40}.$$

(b) Corr(X,Y).

Solution: For the correlation, we need to find $Var(X)$ and $Var(Y)$. To do so we find the second moments of X and Y.

$$\begin{aligned} E[Y^2] &= \int_0^{10}\int_0^x \frac{1}{10x} y^2\, dy\, dx \\ &= \int_0^{10} \frac{x^2}{30}\, dx \\ &= \frac{10^3}{90}. \end{aligned}$$

$$
\begin{aligned}
E[X^2] &= \int_0^{10} \int_0^x \frac{1}{10x} x^2 \, dy \, dx \\
&= \int_0^{10} \frac{x^2}{10} \, dx \\
&= \frac{10^3}{30}.
\end{aligned}
$$

Therefore, the correlation is,

$$
Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} = \frac{\frac{10^3}{60} - \frac{10^4}{800}}{\sqrt{\frac{10^3}{90} - \frac{10^3}{20^2}}\sqrt{\frac{10^3}{30} - \frac{10^4}{20^2}}}
$$

10. Roll two fair dice and denote the outcome (X, Y). Let $Z_1 = X + Y$ and $Z_2 = \max\{X, Y\}$. Find $\text{Corr}(Z_1, Z_2)$.

Solution: First note that $Z_1 \in \{0, 1, 2, 3, 4, 5, 6, 8, 10, 12\}$ and $Z_1 \in \{1, 2, 3, 4, 5, 6\}$. Next, we compute the table of the joint pmf, by writing the joint density in terms of X and Y. That is $P(X = i, Y = j) = \frac{1}{36}$ for all $i, j = 1, ..., 6$. The joint density is as follows,

Z_1

Z_2	2	3	4	5	6	7	8	9	10	11	12
1	$\frac{1}{36}$	0	0	0	0	0	0	0	0	0	0
2	0	$\frac{2}{36}$	$\frac{1}{36}$	0	0	0	0	0	0	0	0
3	0	0	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0	0	0	0	0	0
4	0	0	0	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0	0	0	0
5	0	0	0	0	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0	0
6	0	0	0	0	0	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

For Correlation we need the following

$$
E[Z_1 Z_2], E[Z_1], E[Z_2], E[Z_1]^2, E[Z_2]^2
$$

We sum all the terms to find these values,

$$
\begin{aligned}
E[Z_1 Z_2] &= \sum_{i=2}^{1} 2 \sum j = 1^6 ijp(i,j) \\
&\approx 31.22.
\end{aligned}
$$

$$
\begin{aligned}
E[Z_1] &= \sum_{i=2}^{1} 2 \sum j = 1^6 ip(i,j) \\
&= \sum_{i=2}^{1} 2ip_{Z_1}(i) \\
&= 7.
\end{aligned}
$$

$$
\begin{aligned}
E[Z_2] =& \sum_{i=2}^{1} 2 \sum j = 1^6 i p(i,j) \\
=& \sum_{j=1}^{6} j p_{Z_2}(j) \\
\approx& 4.472.
\end{aligned}
$$

$$
\begin{aligned}
E[Z_1^2] =& \sum_{i=2}^{1} 2 \sum j = 1^6 i^2 p(i,j) \\
=& \sum_{i=2}^{1} 2 i^2 p_{Z_1}(i) \\
=& 54.83.
\end{aligned}
$$

$$
\begin{aligned}
E[Z_2^2] =& \sum_{i=2}^{1} 2 \sum j = 1^6 i p(i,j) \\
=& \sum_{j=1}^{6} j^2 p_{Z_2}(j) \\
\approx& 21.972.
\end{aligned}
$$

This gives the variances and covariances

$$
\begin{aligned}
Cov(Z_1, Z_2) =& E[Z_1 Z_2] - E[Z_1]E[Z_2] \\
=& 31.22 - (7)(4.472).
\end{aligned}
$$

$$
\begin{aligned}
Var(Z_1) =& E[Z_1^2] - E[Z_1]^2 \\
=& 54.83 - 49 \\
\approx&
\end{aligned}
$$

$$
\begin{aligned}
Var(Z_2) =& E[Z_2^2] - E[Z_2]^2 \\
=& 21.972 - (4.472)^2 \\
\approx&
\end{aligned}
$$

$$
Corr(Z_1, Z_2) = \frac{Cov(Z_1, Z_2)}{\sqrt{Var(Z_1)}\sqrt{Var(Z_2)}} \approx ...
$$

11. Let X and Y be Exp(λ) and independent. Let Z be Gamma$(2, \lambda)$ distributed. That is, Z has the density function

$$
f_Z(z) = \begin{cases} \lambda^2 z e^{-\lambda z}, & q \geq 0 \\ 0, & \text{otherwise} \end{cases}
$$

(a) Find the moment generating function of X, $M_X(t)$ and consequently of Y, $M_Y(t)$ (they are the same).

Solution: The moment generating function for X is,

$$\begin{aligned} M_X(t) = E[e^{tX}] &= \int_0^\infty e^{tx}\lambda e^{-\lambda x}\, dx \\ &= \int_0^\infty \lambda e^{(t-\lambda)x}\, dx \\ &= \left(\frac{\lambda e^{(t-\lambda)x}}{t-\lambda}\right)_{x=0}^{\infty}. \end{aligned}$$

The integral converges only if $t < \lambda$. Thus,

$$M_Y(t) = M_X(t) = \begin{cases} \frac{\lambda}{\lambda - t}, & t < \lambda \\ \infty, & t \geq \lambda. \end{cases}$$

(b) Find the moment generating function of Z, $M_Z(t)$.

Solution: To calculate the moment generating function of Z, we integrate by parts,

$$\begin{aligned} M_Z(t) = E[e^{tZ}] &= \int_0^\infty e^{tz}\lambda^2 z e^{-\lambda z}\, dz \\ &= \int_0^\infty \lambda^2 z e^{(t-\lambda)z}\, dz \\ &= \frac{\lambda^2 z}{(t-\lambda)} e^{(t-\lambda)}\Big|_{z=0}^{\infty} - \int_0^\infty \frac{\lambda^2}{(t-\lambda)} e^{(t-\lambda)z}\, dz \\ &= \frac{\lambda^2 z}{(t-\lambda)} e^{(t-\lambda)}\Big|_{z=0}^{\infty} - \frac{\lambda^2}{(t-\lambda)^2} e^{(t-\lambda)z}\Big|_{z=0}^{\infty} \end{aligned}$$

Again, the integral converges only if $t < \lambda$. Then

$$M_Z(t) = \begin{cases} \frac{\lambda^2}{(t-\lambda)^2}, & t < \lambda, \\ \infty, & t \geq \lambda. \end{cases}$$

(c) Using methods from Section 7.3, show that Z and $X + Y$ have the same distribution.

Solution: By fact 7.13 of Section 7.3, for $X + Y$ where X and Y are independent, we have

$$M_{X+Y}(t) = M_X(t)M_Y(t) = \frac{\lambda}{\lambda - t}\frac{\lambda}{\lambda - t} = \frac{\lambda^2}{(t-\lambda)^2} = M_Z(t).$$

This is the moment generating function of a Gamma$(2, \lambda)$ distribution. Therefore, Z and $X + Y$ have the same distribution.