HW #9 Solutions

(1)

(1) 3.77

a) For this problem we use row reduction to find the inverse

$$[A \mid I_2] = \left[\begin{array}{cc|cc} 7 & 4 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{array}\right] \begin{array}{l} R_1/7 \\ R_2/5 \end{array}$$

$$\Rightarrow \left[\begin{array}{cc|cc} 1 & 4/7 & 1/7 & 0 \\ 1 & 3/5 & 0 & 1/5 \end{array}\right] R_2 = R_2 - R_1 \quad \left[\begin{array}{cc|cc} 1 & 4/7 & 1/7 & 0 \\ 0 & 3/5 - 4/7 & -1/7 & 1/5 \end{array}\right]$$

$$R_2 \cdot 35 \quad \left[\begin{array}{cc|cc} 1 & 4/7 & 1/7 & 0 \\ 0 & \underbrace{21 - 20}_{1} & -5 & 7 \end{array}\right] R_1 = R_1 - 4/7 R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1/7 + 20/7 & -4 \\ 0 & 1 & -5 & 7 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} 3 & -4 \\ -5 & 7 \end{bmatrix}$$

b) For b) we use Cramer's rule

$$C = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix} \Rightarrow C^{-1} = \frac{1}{\det(C)} \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$$

$$\det(C) = 4 \cdot 3 - (-2)(-6)$$

$$= 12 - 12 = 0$$

$\therefore$ inverse does not exist!

3.78 c) Now we must use row reduction to find the inverse. (2)

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & 0 & 0 \\ -1 & -1 & 5 & 0 & 1 & 0 \\ 2 & 7 & -4 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 = R_2 + R_1 \\ R_3 = R_3 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 3 & 4 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 - 3R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -6 & -1 & -2 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -5 & -3 & 1 \end{array}\right] \begin{array}{l} R_1 = R_1 + 6R_3 \\ R_2 = R_2 - R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -31 & -20 & 6 \\ 0 & 1 & 0 & 6 & 4 & -1 \\ 0 & 0 & 1 & -5 & -3 & 1 \end{array}\right]$$

Check
$$\begin{bmatrix} -31 & -20 & 6 \\ 6 & 4 & -1 \\ -5 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -4 \end{bmatrix} = \begin{bmatrix} -31+20+12 & 0 & 0 \\ 0 & 12-4-7 & 0 \\ 0 & 0 & 20-15-4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Thus } C^{-1} = \begin{bmatrix} -31 & -20 & 6 \\ 6 & 4 & -1 \\ -5 & -3 & 1 \end{bmatrix}.$$

5.50 a) To solve these we put the vectors into a 3x4 matrix & row reduce. If Rank(A) < 3 then they are linearly dependent.

$$A = \begin{pmatrix} 1 & 2 & -3 & 1 \\ 3 & 7 & 1 & -2 \\ 1 & 3 & 7 & -4 \end{pmatrix} \begin{matrix} R_2 = R_2 - 3R_1 \\ R_3 = R_3 - R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 10 & -5 \\ 0 & 1 & 10 & -5 \end{pmatrix} R_3 = R_3 - R_2 \begin{pmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 10 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ they are <u>linearly dependent</u>.

b)
$$A = \begin{pmatrix} 1 & 3 & 1 & -2 \\ 2 & 5 & -1 & 3 \\ 1 & 3 & 7 & -2 \end{pmatrix} \begin{matrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 1 & -2 \\ 0 & -1 & -3 & 7 \\ 0 & 0 & 6 & 4 \end{pmatrix}$$ Rank(A) = 3 $\therefore$ they are linearly independent

5.68 For each matrix we row reduce A or A^T.

$$A = \begin{pmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{pmatrix} \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \\ R_4 - 2R_1 \end{matrix} \begin{pmatrix} 1 & 3 & -2 & 5 & 4 \\ 0 & 1 & 3 & -2 & 1 \\ 0 & 1 & 4 & -1 & -1 \\ 0 & 1 & 1 & -4 & 5 \end{pmatrix}$$

$$\begin{matrix} R_3 = R_3 - R_2 \\ R_4 = R_4 - R_2 \end{matrix} \begin{pmatrix} 1 & 3 & -2 & 5 & 4 \\ 0 & 1 & 3 & -2 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & -3 & -3 & 6 \end{pmatrix} R_4 = R_4 + 3R_3 \begin{pmatrix} \boxed{1} & 3 & -2 & 5 & 4 \\ 0 & \boxed{1} & 3 & -2 & 1 \\ 0 & 0 & \boxed{1} & \boxed{1} & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus $\text{Rank}(A)=3$.

b) $B=\begin{bmatrix} 1 & 2 & -3 & -2 & -3 \\ 1 & 3 & -2 & 0 & -4 \\ 3 & 8 & -7 & -2 & -11 \\ 2 & 1 & -9 & -10 & -3 \end{bmatrix}$ $R_2=R_2-R_1$, $R_3=R_3-3R_1$, $R_4=R_4-2R_1$

$\begin{pmatrix} 1 & 2 & -3 & -2 & -3 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 2 & 2 & 4 & -2 \\ 0 & -3 & -3 & -6 & 3 \end{pmatrix}$ $R_3=R_3-2R_2$, $R_4=R_4-3R_2$

$\begin{pmatrix} \boxed{1} & 2 & -3 & -2 & -3 \\ 0 & \boxed{1} & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ $\text{Rank}(B)=2$

c) Consider C^T

$C^T=\begin{pmatrix} 1 & 4 & 5 & -1 \\ 1 & 5 & 8 & -2 \\ 2 & 5 & 1 & 2 \end{pmatrix}$ $R_2=R_2-R_1$, $R_3=R_3-2R_1$

$\begin{pmatrix} 1 & 4 & 5 & -1 \\ 0 & 1 & 3 & 3 \\ 0 & -3 & -9 & 4 \end{pmatrix}$ $R_3=R_3+3R_2$

$\begin{pmatrix} \boxed{1} & 4 & 5 & -1 \\ 0 & \boxed{1} & 3 & 3 \\ 0 & 0 & 0 & \boxed{13} \end{pmatrix}$ $\text{Rank}(C^T)=3$

d) Consider D^T

$D^T=\begin{pmatrix} 2 & 3 & -6 & 5 \\ 1 & -7 & 1 & -8 \end{pmatrix}$ $R_1=R_1-2R_2$ $\begin{pmatrix} 0 & \boxed{17} & -8 & 21 \\ \boxed{1} & -7 & 1 & -8 \end{pmatrix}$ $\text{Rank}(D^T)=2$

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10.52

a) $\det(A) = \begin{vmatrix} t-4 & 3 \\ 2 & t-9 \end{vmatrix} = 0$

$$\Rightarrow (t-4)(t-9) - 6 = 0$$

$$t^2 - 13t + 36 - 6 = 0$$

$$t^2 - 13t + 30 = 0$$

$$(t-10)(t-3) = 0$$

$$t = 10, 3$$

b) $\det(A) = \begin{vmatrix} t-1 & 4 \\ 3 & t-2 \end{vmatrix} = 0$

$$\Rightarrow (t-1)(t-2) - 12 = 0$$

$$t^2 - 3t + 2 - 12 = 0$$

$$t^2 - 3t - 10 = 0$$

$$(t-5)(t+2) = 0$$

$$t = 5, -2$$

10.53 b) $B = \begin{pmatrix} 3 & -2 & -4 \\ 2 & 5 & -1 \\ 0 & 6 & 1 \end{pmatrix}$ we use the first column to compute det.

$$\det(B) = 3\begin{vmatrix} 5 & -1 \\ 6 & 1 \end{vmatrix} - 2\begin{vmatrix} -2 & -4 \\ 6 & 1 \end{vmatrix} + 0\cdot\begin{vmatrix} \cdots \end{vmatrix}$$

$$= 3\left(5\cdot 1 - 6\cdot(-1)\right) - 2\left((-2)(1) - (6)(-4)\right)$$

$$= 3(11) - 2(22) = -11$$

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d) $D = \begin{pmatrix} 7 & 6 & 5 \\ 1 & 2 & 1 \\ 3 & -2 & 1 \end{pmatrix}$ we use the first row to find the det

$$\det(D) = 7\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix} - 6\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + 5\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix}$$

$$= 7(2-(-2)) - 6(1-3) + 5(-2-6)$$

$$= 28 + 12 - 40 = \boxed{0}$$

2 a) if $a \neq 0$ we divide both sides by a,

$\therefore$ $\frac{ab}{a} = \frac{ac}{a} \Rightarrow b = c.$

if $a = 0$,

$0 \cdot b = 0$
$0 \cdot c = 0$
$\therefore$ $ab = ac$.

b) $AB = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$

$AC = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$

$\therefore$ $AB = AC$ but $\underbrace{\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}}_{B} \neq \underbrace{\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}}_{C}$

c) $\det(A) = \begin{vmatrix} 1 & 2 \\ -1 & -2 \end{vmatrix} = (-2-(-2)) = 0.$

d) If $Det(D) \neq 0$ then $\exists$ an inverse of D call it D^{-1} such that $DD^{-1} = D^{-1}D = I$. then acting the inverse on the left hand side yields

$$DB = DC$$

$$\Rightarrow D^{-1}DB = D^{-1}DC$$

$$\Rightarrow IB = IC$$

$$\Rightarrow B = C.$$

e) The equivalent of $a \neq 0$ is $det(A) \neq 0$.

3. a) $det(A) \neq 0$

b) $\exists$ a unique soln to $A\vec{x} = \vec{b}$ $\forall$ $\vec{b}$

c) the rows of A are lin indep

d) " cols " " " "

e) $rank(A) = n$

f) A has zero free variables