SM 316 – Spring 2019

Homework 3
Solutions

Due: Monday 04 FEB 2019

PLEASE READ THE INSTRUCTIONS/SUGGESTIONS ON THE COURSE WEBPAGE.

Hand in the following problems:

1. From the text book, 3.8, 3.11, 3.13, 3.14, 3.29

   3.8: Find the probability distribution of the random variable W in Exercise 3.3, assuming that the coin is biased so that a head is twice as likely to occur as a tail. Ex 3.3: Let W be a random variable giving the number of heads minus the number of tails in three tosses of a coin. List the elements of the sample space S for the three tosses of the coin and to each sample point assign a value w of W.

   Solution: *Note that the sample space for W is*

   $$S = \{-3, -1, 1, 3\}$$

   corresponding to 3T, 1H 2T, 2H 1T, and 3H respectively. For 3H, the only outcome is HHH, and since H is twice as likely than T, we have

   $$P(W = 3) = P(HHH) = (2/3)^3.$$

   For $W = 1$ we have the possibilities HHT, HTH, THH. Thus

   $$P(W = 1) = 3(2/3)^2(1/3).$$

   Similarly,

   $$P(W = -1) = 3(2/3)(1/3)^2, \quad P(W = -3) = (1/3)^2.$$

   Thus the pmf for W, $f(w)$, is given by the table,

w	-3	-1	1	3
$f(w)$	0.037	0.222	0.444	0.296

   Note that the pmf sums to 1.

   3.11: A shipment of 7 television sets contains 2 defective sets. A hotel makes a random purchase of 3 of the sets. If x is the number of defective sets purchased by the hotel, find the probability distribution of X. Express the results graphically as a probability histogram.

   Solution: *Note that X can be 0,1,or 2. Let us give an ordering to this set and label the tvs 1 through 7. Then the total number of ways we can buy 3 sets is $7 \cdot 6 \cdot 5$. The number of ways we can purchase 3 non-defective sets would be $5 \cdot 4 \cdot 3$. For 1 defective set it is $2 \cdot 5 \cdot 4$ and* *we can buy the defective set first, second or third, thus giving* $3(2 \cdot 5 \cdot 4)$. *2 defective sets is similar* $3(2 \cdot 1 \cdot 5)$. *Thus the pmf for* X, $f(x)$, *is*

x	0	1	2
$f(x)$	0.286	0.57	0.141

Note that the pmf sums to 1.

3.13: The probability distribution of X, the number of imperfections per 10 meters of a synthetic fabric in continuous rolls of uniform width, is given by

x	0	1	2	3	4
$f(x)$	0.41	0.37	0.16	0.05	0.01

Construct the cumulative distribution function of X.

Solution: *Recall that the CDF for a discrete RV is*

$$F(x) = P(X \leq x) = \sum_{all\ t \leq x} f(t).$$

Thus $F(x) = 0$ *for* $x < 0$, $F(x) = P(X = 0)$ *for* $0 \leq x < 1$, $F(x) = P(X = 0) + P(X = 1)$ *for* $1 \leq x < 2$ *and so on. Thus the CDF is*

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.41 & 0 \leq x < 1 \\ 0.78 & 1 \leq x < 2 \\ 0.94 & 2 \leq x < 3 \\ 0.99 & 3 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$

3.14: The waiting time, in hours, between successive speeders spotted by a radar unit is a continuous random variable X with cumulative distribution function

$$F(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-8x}, & x \geq 0. \end{cases}$$

Find the probability of waiting less than 12 minutes between successive speeders (a) using the cumulative distribution function of X;

Solution: *12 minutes is 0.2 hrs. Using the CDF we want to find* $P(X \leq 0.2)$ *(Note* $P(X \leq 0.2) = P(X < 0.2)$ *since* X *is a continuous RV and* $P(X = 0.2) = 0$*). Also note by the definition of CDF,* $P(X \leq 0.2) = F(0.2)$. *Thus,*

$$P(X \leq 0.2) = F(0.2) = 1 - e^{-8 \cdot 0.2} = 1 - e^{-1.6}$$

(b) using the probability density function of X.

Solution: *First we must find the probability density function of X, $f(x)$. To do so, note that the CDF of a continuous RV is,*

$$F(x) = \int_{-\infty}^{x} f(t)\, dt,$$

where f is the pdf. By the Fundamental Theorem of Calculus we have

$$F'(x) = f(x).$$

Taking the derivative of the specific CDF above gives the pdf,

$$f(x) \begin{cases} 0, & x < 0, \\ 8e^{-8x}, & x \geq 0. \end{cases}$$

Note that the pdf integrates to 1. Now we find the probability by integrating,

$$P(X \leq 0.2) = \int_{-\infty}^{0.2} f(x)\, dx = \int_{0}^{0.2} 8e^{-8x}\, dx = \left(-e^{-8x}\right)_{0}^{0.2} = 1 - e^{-1.6}.$$

3.29: An important factor in solid missile fuel is the particle size distribution. Significant problems occur if the particle sizes are too large. From production data in the past, it has been determined that the particle size (in micrometers) distribution is characterized by

$$f(x) \begin{cases} 3x^{-4}, & x > 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Verify that this is a valid density function.

Solution: *To verify it is a valid density function, we take the integral and see that it is one. I.e.*

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{1}^{\infty} 3x^{-4}\, dx = \left(-x^{-3}\right)_{1}^{\infty} = -0 - (-1) = 1.$$

(b) Evaluate $F(x)$.

Solution: *To evaluate $F(x)$ we use the definition of CDF of a continuous RV,*

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt.$$

Because f is piecewise, we have

$$F(x) = \begin{cases} 1 - x^{-3}, & x > 1, \\ 0, & \textit{elsewhere.} \end{cases}$$

(c) What is the probability that a random particle from the manufactured fuel exceeds 4 micrometers?

Solution: *We want to find $P(X > 4) = 1 - P(X \leq 4)$. This can be done quickly with the CDF. That is,*

$$P(X > 4) = 1 - P(X \leq 4) = 1 - F(4) = 1 - (1 - 4^{-3}) = 4^{-3}.$$

2. When a certain basketball player takes his first shot in a game he succeeds with probability 1/2. If he misses his first shot, he loses confidence and his second shot will go in with probability 1/3. If he misses his first 2 shots then his third shot will go in with probability 1/4. His success probability goes down further to 1/5 after he misses his first 3 shots. If he misses his first 4 shots then the coach will remove him from the game. Assume that the player keeps shooting until he succeeds or he is removed from the game. Let X denote the number of shots he misses until his first success or until he is removed from the game. Calculate the probability mass function of X.

Solution: *Note that the probabilities in the problem statement are conditional probabilities. Note that X can be $\{0, 1, 2, 3, 4\}$. The probability that $X = 0$ is $1/2$. For the rest of them, we note that*

$$P(X = 1) = P(\text{misses 1st shot AND makes 2nd shot})$$

$$= P(\text{misses 1st shot})P(\text{makes 2nd shot}|\text{misses 1st shot}) = \frac{1}{2} \cdot \frac{1}{3}.$$

For $X = 2$ we have

$$P(X = 2) = P(\text{misses 1st shot AND misses 2nd shot AND makes 3rd shot})$$

$$= P(\text{misses 1st shot AND 2nd shots})P(\text{makes 3rd shot}|\text{misses 1st and 2nd shots})$$

and using the first part of $P(X = 1)$ we have,

$$= P(\text{misses 1st shot})P(\text{misses 2nd shot}|\text{misses 1st shot})P(\text{makes 3rd shot}|\text{misses 1st and 2nd shots})$$

$$= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{4}.$$

This pattern is continued to get

$$P(X = 3) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{1}{5},$$

and

$$P(X = 4) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5}.$$

Thus the pmf of X, $f(x)$, is

x	0	1	2	3	4
$f(x)$	0.5	0.167	0.083	0.05	0.20

Note that the pmf sums to 1.

3. Let X be the number of coin flips needed until you see the first tails. Find the probability mass function, and cumulative distribution function. Graph the CDF.

Solution: *The pmf of X is found by considering the $P(X = n)$. If $X = n$, then the first $n - 1$ flips must all be heads. This happens with probability $(1/2)^{n-1}$. Then the nth flip is Tails which happens with probability $1/2$. Thus the pmf is*

$$P(X = n) = \left(\frac{1}{2}\right)^n.$$

For the CDF, we use the definition for a discrete RV,

$$F(x) = P(X \le x) = \sum_{\textit{all } t \le x} f(t).$$

Let $x = n$ *a natural number. Then we have*

$$F(n) = P(X \le n) = P(X=1)+P(X=2)+P(X=3)+\cdots+P(X=n) = \left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n}\right),$$

$$= \frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}\right).$$

Note that the final sum is a finite geometric series with ratio $r = 1/2$. *Using Wikipedia, we have the formula*

$$\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}\right) = \frac{1}{2}\left(\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}\right).$$

Thus the CDF for X *is*

$$F(x) = \begin{cases} 0, & x < 1 \\ 1 - \frac{1}{2^n} & n \le x < n+1 \end{cases}$$

for $n = 1, 2, 3, \ldots.$